

# MODULI SPACES OF VECTOR BUNDLES WITH FIXED DETERMINANT OVER A REAL CURVE

THOMAS JOHN BAIRD

ABSTRACT. Let  $(\Sigma, \tau)$  denote a Riemann surface of genus  $g \geq 2$  equipped with an anti-holomorphic involution  $\tau$ . In this paper we study the topology of the moduli space  $M(r, \xi)^\tau$  of stable Real vector bundles over  $(\Sigma, \tau)$  of rank  $r$  and fixed determinant  $\xi$  of degree coprime to  $r$ .

We prove that  $M(r, \xi)^\tau$  is an orientable and monotone Lagrangian submanifold of the complex moduli space  $M(r, \xi)$  so it determines an object in the appropriate Fukaya category. We derive recursive formulas for the  $\mathbb{Z}_2$ -Betti numbers of  $M(r, \xi)^\tau$  and compute  $\mathbb{Z}_p$ -Betti numbers for odd  $p$  through a range of degrees. We deduce that if  $r$  is even and  $g \gg 0$ , then  $M(r, \xi)^\tau$  and  $M(r, \xi')^\tau$  have non-isomorphic cohomology groups unless  $\xi$  and  $\xi'$  have equivalent Stiefel-Whitney classes modulo automorphisms of  $(\Sigma, \tau)$ . If  $r$  is even, and  $g \gg 0$  is even, we prove that the Betti numbers of  $M(r, \xi)^\tau$  distinguish topological types of  $(\Sigma, \tau; \xi)$ . If  $r = 2$  and  $g$  is odd, we compute all  $\mathbb{Z}_p$ -Betti numbers of  $M(2, \xi)^\tau$ .

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## 1. INTRODUCTION

Let  $\Sigma$  denote a Riemann surface of genus  $g \geq 2$  and let  $M(r, d)$  the moduli space of semi-stable holomorphic vector bundles over  $\Sigma$  of rank  $r$  and degree  $d$ . For simplicity, we assume throughout this introduction that  $r$  and  $d$  are coprime, which implies that  $M(r, d)$  is a non-singular projective variety (we relax this condition in the rest of the paper). Given  $\xi \in \text{Pic}_d(\Sigma) = M(1, d)$ , denote by  $M(r, \xi)$  the moduli space of rank  $r$  bundles with fixed determinant  $\xi$ . We may regard  $M(r, \xi)$  as a fibre  $\det^{-1}(\xi)$  of the fibre bundle

$$(1.1) \quad \det : M(r, d) \rightarrow M(1, d)$$

which sends the isomorphism class  $[\mathcal{E}]$  to  $[\wedge^r \mathcal{E}]$ . A line bundle  $\eta \in \text{Jac}(\Sigma) = M(1, 0)$ , determines an isomorphism  $M(r, \xi) \cong M(r, \xi \otimes \eta^r)$ . In particular, the subgroup  $\text{Jac}[r] \leq \text{Jac}(\Sigma)$ , of  $r$ -th roots of unity acts naturally on  $M(r, \xi)$ . Tensor product determines an isomorphism

$$M(r, d) \cong M(r, \xi) \times_{\text{Jac}[r]} M(1, 0),$$

where the right side is a the mixed quotient with respect to tensor product actions on both factors.

In [1], Atiyah and Bott calculated the cohomology groups of  $M(r, d)$  and  $M(r, \xi)$ . In particular they proved that:

- (1) Both  $H^*(M(r, d); \mathbb{Z})$  and  $H^*(M(r, \xi); \mathbb{Z})$  are torsion free.
- (2) The action of  $Jac[r]$  on  $H^*(M(r, \xi); \mathbb{Z})$  is trivial.
- (3)  $H^*(M(r, d); \mathbb{Z}) \cong H^*(M(r, \xi); \mathbb{Z}) \otimes H^*(M(1, d); \mathbb{Z})$ .

Indeed, (3) follows from (1) and (2) by a simple argument. One of the goals of the present paper is to explore to what degree these properties hold when  $M(r, d)$  is replaced by a moduli space of Real bundles over a real curve, as defined in [7, 15].

A *real curve*  $(\Sigma, \tau)$  is a Riemann surface  $\Sigma$  equipped with an antiholomorphic involution  $\tau$ . The fixed point set  $\Sigma^\tau$  is a union of circles, called the *real circles* of  $(\Sigma, \tau)$ . There is an induced antiholomorphic involution on  $M(r, d)$  (which we also denote  $\tau$ ), defined by

$$\tau([E]) = [\tau^* \overline{E}].$$

If  $\xi \in M(1, d)$  is fixed by  $\tau$ , then  $\tau$  restricts to an involution on  $M(r, \xi)$ , which we also denote  $\tau$ . The fixed point sets by  $M(r, d)^\tau$  and  $M(r, \xi)^\tau$  are half dimensional real submanifolds of  $M(r, d)$  and  $M(r, \xi)$  respectively.

If  $\Sigma^\tau$  has  $a > 0$  path components, then  $M(r, d)^\tau$  has  $2^{a-1}$  path components, parametrized by cohomology classes  $w \in H^1(\Sigma^\tau; \mathbb{Z}_2)$  which satisfy

$$(1.2) \quad w(\Sigma^\tau) \equiv d \pmod{2}.$$

Denote by  $M(r, d)_w^\tau$  the path component corresponding to  $w$ . The holomorphic bundles  $\mathcal{E} \in M(r, d)^\tau$  are precisely those that admit an anti-holomorphic bundle automorphism  $\tilde{\tau}$  lifting  $\tau$  and we call such  $\mathcal{E}$  (*holomorphic*) *Real vector bundles*. The invariant  $w$  is simply the first Stiefel-Whitney class of the real locus  $E^\tau \rightarrow \Sigma^\tau$ . We call a real circle  $S \subseteq \Sigma^\tau$  *odd* (resp. *even*) with respect to  $E \in M(r, d)_w^\tau$  if  $w(S) = 1$  (resp.  $w(S) = 0$ ).

If  $\Sigma^\tau$  is empty,  $M(r, d)_0^\tau$  exists as before whenever  $d$  is even, but there may also be a path component  $M(r, d)_{\mathbb{H}}^\tau$  corresponding to what are called Quaternionic vector bundles (see [7] for a details). However tensoring with a Quaternionic line bundle of degree  $d'$  determines isomorphism  $M(r, d)_{\mathbb{H}}^\tau \cong M(r, d + rd')_0^\tau$ , so these Quaternionic vector bundles can safely be neglected for our purposes.

As a Lie group,  $M(1, 0)_0^\tau \cong U(1)^g$ . Let  $\Gamma_g \cong (\mathbb{Z}/r)^g$  be the  $r$ -torsion subgroup of  $M(1, 0)_0^\tau$ . Note that  $\Gamma_g$  is a subgroup of  $Jac[r] \cap M(1, 0)^\tau$ . We have an isomorphism

$$M(r, d)_w^\tau \cong M(r, \xi)^\tau \times_{\Gamma_g} M(1, 0)_0^\tau$$

where  $\xi \in M(1, d)_w^\tau$  and the right side is a the mixed quotient with respect to tensor product actions on both factors (see [4], §6). Our first main theorem is a version of (2) and (3) for mod 2 coefficients.

**Theorem 1.1.** *The action of  $\Gamma_g$  on  $H^*(M(r, \xi); \mathbb{Z}_2)$  is trivial and we have an isomorphism*

$$H^*(M(r, d)_w^\tau; \mathbb{Z}_2) \cong H^*(M(r, \xi)^\tau; \mathbb{Z}_2) \otimes H^*(M(1, 0)_0^\tau; \mathbb{Z}_2).$$

Since recursive formulas for the mod 2 Betti numbers of  $M(r, \xi)^\tau$  were computed in [2, 11], Theorem 1.1 yields formulas for the Betti numbers of  $M(r, \xi)^\tau$ . We present explicit formulas for  $r = 2$  and  $r = 3$  in §6 for convenience.

One of the peculiarities of the mod 2 Betti numbers formulas [2, 11] is that they depend only on the real curve  $(\Sigma, \tau)$  and not on the Stiefel-Whitney class  $w$ . When  $r$  is odd this can be explained by the existence of homeomorphisms  $M(r, \xi)^\tau \cong M(r, \xi')^\tau$  for any pair of Real line bundles  $\xi$  and  $\xi'$ , determined by tensoring with third Real line bundle  $\eta$  such that  $\xi \cong \xi' \otimes \eta^r$ . However, such homeomorphisms generally do not exist when  $r$  is even.

**Theorem 1.2.** *For  $i \in \{1, 2\}$  let  $(\Sigma_i, \tau_i)$  be real curves of genus  $g_i$  and  $a_i$  real circles, and let  $\xi_i$  be Real line bundles over  $(\Sigma_i, \tau_i)$  with  $c_i$  many even circles. Then we have an isomorphism of graded groups*

$$(1.3) \quad H^*(M(r, \xi_1)^{\tau_1}; \mathbb{Z}) \cong H^*(M(r, \xi_2)^{\tau_2}; \mathbb{Z})$$

only if  $g_1 = g_2$ ,  $a_1 = a_2$ .

Suppose additionally that  $r$  is even and that either  $r = 2$  and  $g_1 \geq 5$  or  $r \geq 4$  and  $g_1 \geq 3$ . Then (1.3) holds only if  $c_1 = c_2$ .

Suppose in further addition that  $g$  is even and  $g \geq 6$ . Then (1.3) holds only if there exists a homeomorphism  $f : \Sigma_1 \rightarrow \Sigma_2$  such that  $f \circ \tau_1 = \tau_2 \circ f$ .

Theorem 1.2 is proven by computing the odd characteristic Betti numbers in all degrees less than  $g(r-1) - 1$ . For rank 2 bundles and odd genus, we can do better and compute the entire Poincaré polynomial.

**Theorem 1.3.** *Let  $(\Sigma, \tau)$  be a real curve of odd genus and let  $\xi$  be a Real line bundle of odd degree for which  $\xi$  has  $c$  even circles. For any field  $\mathbb{F}$  of characteristic  $\neq 2$  we have*

$$(1.4) \quad P_t(M(2, \xi)^\tau; \mathbb{F}) = \frac{1}{2}(1+t^3)^{g-c-1}((1+t)^c(1+t^2)^c + (1-t)^c(1-t^2)^c).$$

Note that the Betti number formula (1.4) fails to fully distinguish between topological types of real curves, in contrast to what happens when  $g$  is even. In particular, if for  $i \in \{1, 2\}$ ,  $(\Sigma_i, \tau_i, \xi_i)$  are real curves with the same odd genus  $g$ , the same number of real circles  $a$ , equipped with Real line bundles with the same number of even circles  $c$ , but  $\Sigma_1 \setminus \Sigma_1^{\tau_1}$  is connected and  $\Sigma_2 \setminus \Sigma_2^{\tau_2}$  is disconnected, then the corresponding moduli spaces  $M(2, \xi_1)^{\tau_1}$  and  $M(2, \xi_2)^{\tau_2}$  have identical Betti numbers in all characteristics. This is a peculiar fact for which I have no moral explanation.

In [4] the Poincaré polynomial of the invariant subring  $H^*(M(2, \xi)^\tau; \mathbb{Q})^{\Gamma_2}$  was shown to equal  $(1+t^3)^{g-1}$ , when  $c \neq 0$ . We deduce that the real analogue of (2) is false.

**Corollary 1.4.** *Let  $\mathbb{F}$  be a field of characteristic  $\neq 2$ . Then the action of  $\Gamma_g$  on  $H^*(M(2, \xi)^\tau; \mathbb{F})$  is non-trivial in general.*

We also calculate the fundamental group of  $M(r, \xi)^\tau$  except when  $r = 2$  and  $g = 2$ . A consequence is that the real analogue of (1) is false.

**Theorem 1.5.** *Let  $(\Sigma, \tau)$  denote a real curve of genus  $g \geq 2$  with  $a$  real circles, let  $\xi \in M(1, d)^\tau$  with  $b$  odd circles, and let  $r \geq 2$ . We have an isomorphism*

$$\pi_1(M(r, \xi)^\tau) \cong \begin{cases} \mathbb{Z}/2 \ltimes (\mathbb{Z}/2)^a & \text{if } r \geq 3 \\ \mathbb{Z}/2 \ltimes ((\mathbb{Z}/2)^b \times (\mathbb{Z})^{a-b}) & \text{if } r = 2. \end{cases}$$

where  $\mathbb{Z}/2$  acts diagonally: trivially on the  $\mathbb{Z}/2$  factors and by  $-1$  on the  $\mathbb{Z}$  factors.

Consequently,  $H_1(M(r, \xi)^\tau; \mathbb{Z}) \cong (\mathbb{Z}/2)^a$  unless  $g = 2$  and  $r = 2$ .

Note that the conclusion of Theorem 1.5 does not extend to the case  $g = 2$  and  $r = 2$ . This case is completely worked out in [5] where in some examples  $H_*(M(r, \xi)^\tau; \mathbb{Z})$  is torsion-free.

The strategy for proving all of the above results is to study the real Harder-Narasimhan stratification introduced in [11, 2], which is a real analogue of the complex Harder-Narasimhan stratification studied by Atiyah and Bott [1]. This stratification relates the topology of  $M(r, d)^\tau$  with the topology of a group of real gauge transformations,  $\mathcal{G}_{\mathbb{R}}$ , and was used in [11, 2] to compute  $\mathbb{Z}_2$ -Betti numbers of  $M(r, d)^\tau$  and in [4] to compute the  $\mathbb{Q}$ -Betti numbers of  $M(2, d)^\tau$ . In similar fashion, we relate the topology of  $M(r, \xi)^\tau$  with  $C\mathcal{G}_{\mathbb{R}}$ , the group of real gauge transformations with constant determinant.

One motivation for studying these fixed determinant moduli spaces is that they form a rich and geometrically interesting class of real Lagrangian submanifolds of  $M(r, \xi)$  endowed with the standard Atiyah-Bott symplectic form. In §7, we prove a couple results that ensure these  $M(r, \xi)^\tau$  have well-defined Lagrangian Floer cohomology over  $\mathbb{Z}_2$ -coefficients and thus determine objects in the appropriate Fukaya category [6, 9].

**Theorem 1.6.**  *$M(r, \xi)$  is orientable and monotone with minimal Maslov index a positive multiple of two.*

We have not been able to prove that  $M(r, \xi)^\tau$  is relatively spin in general, so the Fukaya category is defined only with  $\mathbb{Z}_2$  coefficients. However in [5] we prove that when  $r = 2$ ,  $M(2, \xi)^\tau$  is relatively spin in  $M(2, \xi)$  so it determines an object in a  $\mathbb{Z}$ -Fukaya category.

We summarize the contents of the paper. In §2 we outline the basic strategy relating  $M(r, \xi)$  to the real Harder-Narasimhan filtration and the classifying space of the real gauge group  $C\mathcal{G}_{\mathbb{R}}$ . The technical heart of the paper is §3 where we compute the Betti numbers of  $BC\mathcal{G}_{\mathbb{R}}$  using an Eilenberg-Moore spectral sequence and also compute the fundamental group, yielding proofs of Theorems 1.2 and 1.5. In §4 we prove that the real Harder-Narasimhan filtration is  $C\mathcal{G}_{\mathbb{R}}$ -equivariantly perfect with respect to mod 2 coefficients, completing the proof of Theorem 1.1. In §5 we prove Theorem 1.3 by showing that the Thom spaces of the unstable strata are  $\mathbb{F}$ -acyclic. In the remaining sections we illustrate our results with some examples and prove Theorem 1.6.

**Notational conventions:** If  $G$  is a topological group acting on a topological space  $X$  we denote  $X_{hG} = EG \times_G X$  the homotopy quotient. Denote the Poincaré series  $P_t(X; \mathbb{F}) = \sum_{i=0}^{\infty} \dim(H^i(X; \mathbb{F}))t^i$ .

## 2. BASIC STRATEGY

We recall the construction of  $M(r, d)_w^\tau$  from [7, 15]. We no longer require  $\gcd(r, d) = 1$ .

Fix a real curve  $(\Sigma, \tau)$ . Topologically, real curves are classified (see [17]) by invariants  $(g, a, \epsilon)$  where  $g \geq 0$  is the genus of  $\Sigma$ ,  $a = \pi_0(\Sigma^\tau)$  is the number of real circles, and  $\epsilon = 1$  if  $\Sigma \setminus \Sigma^\tau$  is connected and  $\epsilon = 0$  if  $\Sigma \setminus \Sigma^\tau$  is disconnected. A real curve with invariants  $(g, a, \epsilon)$  exists if and only if  $1 - \epsilon \leq a \leq g + 1 - \epsilon$  and  $g + 1 \equiv a \pmod{2}$  if  $\epsilon = 0$ .

Fix a smooth complex vector bundle  $\pi : E \rightarrow \Sigma$  of rank  $r$  and degree  $d$  endowed with an anti-linear bundle  $\tilde{\tau} : E \rightarrow E$  such that  $\pi \circ \tilde{\tau} = \tau \circ \pi$ . We call  $(E, \tilde{\tau})$  a  $C^\infty$ -Real vector bundle over  $(\Sigma, \tau)$ . The fixed point set  $E^{\tilde{\tau}} \rightarrow \Sigma^\tau$  is a  $\mathbb{R}^r$ -bundle and we require that  $w = w_1(E^{\tilde{\tau}})$ . Topologically,  $(E, \tilde{\tau})$  is classified ([7]) by  $d$  and  $w$ , subject to the condition that

$$(2.1) \quad d \equiv w(\Sigma^\tau) \pmod{2}.$$

and  $w(\Sigma^\tau)$  is equal the number of odd circles for  $w$  (see [7]).

Denote by  $\mathcal{C} = \mathcal{C}(E) = \mathcal{C}(r, d)$  (the Sobolev completion of) the space holomorphic structures on  $E$ , represented by  $L_s^2$ -Cauchy-Riemann operators  $\bar{\partial}$  on  $E$  for some fixed  $s > 1$ . Denote by  $\mathcal{C}^{\tilde{\tau}}$  the subspace of holomorphic structures that commute with  $\tilde{\tau}$ , which we call Real holomorphic structures. As topological spaces both  $\mathcal{C}$  and  $\mathcal{C}^{\tilde{\tau}}$  are contractible Banach manifolds.  $\mathcal{C}^{\tilde{\tau}}$  is acted upon by the *real gauge group*

$$\mathcal{G}_{\mathbb{R}} = \mathcal{G}(E)_{\mathbb{R}} = \mathcal{G}^{\tilde{\tau}},$$

consisting of  $L_{s+1}^2$ -gauge transformations that commute with  $\tilde{\tau}$ .

In case  $E = L$  is a line bundle, there is a natural isomorphism  $\mathcal{G}(L) \cong \text{Maps}(\Sigma, \mathbb{C}^*)$ , so the isomorphism type of  $\mathcal{G}(L)$  is independent of  $L$ . If  $(L, \tilde{\tau})$  is a Real line bundle over  $(\Sigma, \tau)$ , then  $\mathcal{G}(L)^{\tilde{\tau}}$  is identified with maps that are equivariant with respect to involutions on  $\Sigma$  and  $\mathbb{C}^*$ , so  $\mathcal{G}(L)^{\tilde{\tau}} = \text{Maps}_{\mathbb{Z}/2}(\Sigma, \mathbb{C})$  is also independent of  $(L, \tilde{\tau})$ . We write

$$\mathcal{G}(1)_{\mathbb{R}} = \mathcal{G}(L)^{\tilde{\tau}}$$

to make this independence explicit.

$\mathcal{C}^{\tilde{\tau}}$  admits a  $\mathcal{G}_{\mathbb{R}}^{\tilde{\tau}}$ -equivariant stratification  $\bigcup_{\mu} \mathcal{C}_{\mu}^{\tilde{\tau}}$  according to real Harder-Narasimhan type ([2] §2). This stratification is equivariantly perfect with respect to the  $\mathcal{G}_{\mathbb{R}}$ -action and  $\mathbb{Z}_2$ -coefficients. This means that

$$P_t(\mathcal{C}_{h\mathcal{G}_{\mathbb{R}}}^{\tilde{\tau}}; \mathbb{Z}_2) = \sum_{\mu} t^{d_{\mu}} P_t((\mathcal{C}_{\mu}^{\tilde{\tau}})_{h\mathcal{G}_{\mathbb{R}}}; \mathbb{Z}_2)$$

where  $d_{\mu}$  is the codimension of  $\mathcal{C}_{\mu}$  in  $\mathcal{C}$ . Since the central subgroup of scalars  $\mathbb{R}^* \leq \mathcal{G}_{\mathbb{R}}$  acts trivially it is sometimes preferable to work with the quotient group  $\bar{\mathcal{G}}_{\mathbb{R}} = \mathcal{G}_{\mathbb{R}}/\mathbb{R}^*$

which acts effectively. The stratum  $\mathcal{C}_{ss}^{\tilde{\tau}}$  consisting of those Real holomorphic structures that are geometrically semistable is dense and open. The  $\mathcal{G}_{\mathbb{R}}$ -action restricts to  $\mathcal{C}_{ss}^{\tilde{\tau}}$  with orbit space

$$\mathcal{C}_{ss}^{\tilde{\tau}}/\mathcal{G}_{\mathbb{R}} = \mathcal{C}_{ss}^{\tilde{\tau}}/\overline{\mathcal{G}}_{\mathbb{R}} = M(r, d)_w^{\tau}.$$

If  $\gcd(r, d) = 1$ , then  $\overline{\mathcal{G}}_{\mathbb{R}}$  acts freely on  $\mathcal{C}_{ss}^{\tilde{\tau}}$  and the quotient exact sequence  $1 \rightarrow \mathbb{R}^* \rightarrow \mathcal{G}_{\mathbb{R}} \rightarrow \overline{\mathcal{G}}_{\mathbb{R}} \rightarrow 1$  splits ([2], Lemma 7.1), so we have a non-canonical isomorphism

$$(2.2) \quad \mathcal{G}_{\mathbb{R}} \cong \overline{\mathcal{G}}_{\mathbb{R}} \times \mathbb{R}^*.$$

Consider now the subgroup  $C\mathcal{G}_{\mathbb{R}} \leq \mathcal{G}_{\mathbb{R}}$  of real gauge transformations with constant determinant. These are the gauge transformations of  $E$  that act as a constant scalar multiplication on the determinant line bundle  $\Lambda^r E$ , so  $C\mathcal{G}_{\mathbb{R}}$  fits into a short exact sequence

$$(2.3) \quad 1 \rightarrow C\mathcal{G}_{\mathbb{R}} \rightarrow \mathcal{G}_{\mathbb{R}} \rightarrow \overline{\mathcal{G}}(1)_{\mathbb{R}} \rightarrow 1,$$

where surjectivity of  $\mathcal{G}_{\mathbb{R}} \rightarrow \overline{\mathcal{G}}(1)_{\mathbb{R}} = \overline{\mathcal{G}(\Lambda^r E)}_{\mathbb{R}}$  follows by considering a Whitney sum decomposition of  $E$  into Real line bundles (see (3.2)). We will later need the following.

**Lemma 2.1.** *The group of path components  $\pi_0(\overline{\mathcal{G}}(1)_{\mathbb{R}})$  is isomorphic to  $\mathbb{Z}^g$  and the identity component of  $\overline{\mathcal{G}}(1)_{\mathbb{R}}$  is contractible. Therefore  $B\overline{\mathcal{G}}(1)_{\mathbb{R}} \cong (S^1)^g$ .*

*Proof.* Since  $\overline{\mathcal{G}}(1)_{\mathbb{R}}$  acts freely on the contractible space  $C(L)^{\tilde{\tau}} = C(L)_{ss}^{\tilde{\tau}}$  it follows that

$$B\overline{\mathcal{G}}(1)_{\mathbb{R}} = C(L)^{\tilde{\tau}}/\overline{\mathcal{G}}(1)_{\mathbb{R}} = M(1, 0)_w^{\tau}.$$

Since  $M(1, 0)_w^{\tau}$  is homeomorphic to  $(S^1)^g$  it follows that  $B\overline{\mathcal{G}}(1)_{\mathbb{R}}$  is a  $K(\mathbb{Z}^g, 1)$  and thus that the quotient map  $\overline{\mathcal{G}}(1)_{\mathbb{R}} \rightarrow \pi_0(\overline{\mathcal{G}}(1)_{\mathbb{R}}) \cong \mathbb{Z}^g$  is a homotopy equivalence.  $\square$

The scalar transformations are contained in  $C\mathcal{G}_{\mathbb{R}}$  so defining  $\overline{C\mathcal{G}}_{\mathbb{R}} = C\mathcal{G}_{\mathbb{R}}/\mathbb{R}^*$ , gives rise to a short exact sequence

$$1 \rightarrow \overline{C\mathcal{G}}_{\mathbb{R}} \rightarrow \overline{\mathcal{G}}_{\mathbb{R}} \rightarrow \overline{\mathcal{G}}(1)_{\mathbb{R}} \rightarrow 1.$$

If  $\gcd(r, d) = 1$  we have with a non-canonical isomorphism

$$(2.4) \quad C\mathcal{G}_{\mathbb{R}} \cong \overline{C\mathcal{G}}_{\mathbb{R}} \times \mathbb{R}^*.$$

**Lemma 2.2.** *Let  $(E, \tilde{\tau})$  be a Real  $C^\infty$ -vector bundle of rank  $r$  and degree  $d$  with  $w := w_1(E^{\tilde{\tau}})$ , and let  $\xi \in M(1, d)_w^{\tau}$ . Then there is a homotopy equivalence  $M(r, \xi) \cong \mathcal{C}_{ss}^{\tilde{\tau}}/C\mathcal{G}_{\mathbb{R}} \cong \mathcal{C}_{ss}^{\tilde{\tau}}/\overline{C\mathcal{G}}_{\mathbb{R}}$ .*

*Proof.* Consider the determinant map  $\det : \mathcal{C}^{\tilde{\tau}} \rightarrow \mathcal{C}(\Lambda^r E)^{\tilde{\tau}}$ . This is equivariant with respect to  $\overline{\mathcal{G}}_{\mathbb{R}}$  and  $\overline{C\mathcal{G}}_{\mathbb{R}}$  is the stabilizer of every point in  $\mathcal{C}(\Lambda^r E)^{\tilde{\tau}}$ . Consequently, we can identify  $(\mathcal{C}_{ss}^{\tilde{\tau}})/\overline{C\mathcal{G}}_{\mathbb{R}}$  as the pull-back of the diagram

$$\begin{array}{c}
 \mathcal{C}_{ss}^{\tilde{\tau}}/\overline{\mathcal{G}}_{\mathbb{R}} = M(r, d)_w^{\tau} \\
 \downarrow \\
 \mathcal{C}(\Lambda^r E)^{\tilde{\tau}} \longrightarrow \mathcal{C}(\Lambda^r E)^{\tilde{\tau}}/\overline{\mathcal{G}}_{\mathbb{R}} = M(1, d)_w^{\tau}
 \end{array}$$

Since both morphisms in the diagram are fibre bundles, the pull-back is homotopy equivalent to the homotopy pull-back. Since  $\mathcal{C}(\Lambda^r E)^{\tilde{\tau}}$  is contractible, we conclude that  $\mathcal{C}_{ss}^{\tilde{\tau}}/\overline{\mathcal{G}}_{\mathbb{R}}$  is homotopy equivalent to the fibre of the determinant map  $M(r, d)_w^{\tau} \rightarrow M(1, d)_w^{\tau}$ .  $\square$

**Corollary 2.3.** *With notation as in Lemma 2.2, if  $\gcd(r, d) = 1$  then we have a homotopy equivalences  $M(r, \xi) \cong (C_{ss})_{h\overline{\mathcal{G}}_{\mathbb{R}}}$  and  $M(r, \xi) \times B\mathbb{R}^* \cong (C_{ss})_{h\mathcal{G}_{\mathbb{R}}}$ .*

*Proof.* If  $\gcd(r, d) = 1$  then  $C\mathcal{G}_{\mathbb{R}} \cong \overline{C\mathcal{G}}_{\mathbb{R}} \times \mathbb{R}^*$  where  $\mathbb{R}^*$  acts trivially and  $\overline{C\mathcal{G}}_{\mathbb{R}}$  acts freely. The result now follows from Lemma 2.2.  $\square$

The strategy for proving Theorem 1.1 is as follows. We have diagram of homotopy quotients

$$\begin{array}{ccc}
 (\mathcal{C}_{ss}^{\tilde{\tau}})_{h\overline{\mathcal{G}}_{\mathbb{R}}} & \longrightarrow & (\mathcal{C}^{\tilde{\tau}})_{h\overline{\mathcal{G}}_{\mathbb{R}}} \\
 \downarrow & & \downarrow \\
 (\mathcal{C}_{ss}^{\tilde{\tau}})_{h\mathcal{G}_{\mathbb{R}}} & \longrightarrow & (\mathcal{C}^{\tilde{\tau}})_{h\mathcal{G}_{\mathbb{R}}}
 \end{array}$$

where arrows are induced by inclusions  $\mathcal{C}_{ss}^{\tilde{\tau}} \hookrightarrow \mathcal{C}^{\tilde{\tau}}$  and  $\overline{\mathcal{G}}_{\mathbb{R}} \hookrightarrow \mathcal{G}_{\mathbb{R}}$ . If  $\gcd(r, d) = 1$ , then this diagram is equivalent up to homotopy to

$$\begin{array}{ccc}
 M(r, \xi)^{\tau} & \longrightarrow & B\overline{C\mathcal{G}}_{\mathbb{R}} \\
 \downarrow i & & \downarrow \\
 M(r, d)_w^{\tau} & \longrightarrow & B\overline{\mathcal{G}}_{\mathbb{R}}.
 \end{array}
 \tag{2.5}$$

Here  $i$  can be identified with the fibre inclusion (1.1). We will show that all of the maps in (2.5) induce  $\mathbb{Z}_2$ -cohomology surjections. Theorem 1.1 then follows by the Leray-Hirsch Theorem.

To prove our results on odd characteristic cohomology, we use the following.

**Corollary 2.4.** *If  $\gcd(r, d) = 1$  then the map (2.5)  $M(r, \xi) \rightarrow B\overline{C\mathcal{G}}_{\mathbb{R}}$  induces a surjection on  $\pi_k$  for  $k \leq g(r-1) - 1$  and an isomorphism for  $k \leq g(r-1) - 2$ . Consequently*

$$H_k(M(r, \xi); \mathbb{Z}) \rightarrow H_k(B\mathcal{G}_{\mathbb{R}}; \mathbb{Z})$$

and

$$H_k(B\mathcal{G}_{\mathbb{R}}; \mathbb{F}) \rightarrow H_k(M(r, \xi); \mathbb{F})$$

are isomorphisms for all  $k \leq g(r-1) - 2$  and coefficient fields  $\mathbb{F}$ .

*Proof.* The codimension of all unstable strata is always greater than or equal to  $g(r-1)$  (an easy exercise given the codimension formula (2.4) in [2]). Therefore the induced map

$$(\mathcal{C}_{ss}^{\tilde{\tau}})_{h\overline{C\mathcal{G}}_{\mathbb{R}}} \rightarrow (\mathcal{C}^{\tilde{\tau}})_{h\overline{C\mathcal{G}}_{\mathbb{R}}} \cong B\overline{C\mathcal{G}}_{\mathbb{R}}$$

must be a surjection on  $\pi_k$  for  $k = g(r-1) - 1$  and an isomorphism for  $k \leq g(r-1) - 2$ . The result follows from Corollary 2.3, the Hurewicz Theorem, and the Universal Coefficient Theorem.  $\square$

### 3. TOPOLOGY OF $BC\mathcal{G}_{\mathbb{R}}$

Let  $\mathcal{G}_{\mathbb{R}} = \mathcal{G}(E)_{\mathbb{R}}$ . In this section, we compute the Betti numbers of  $BC\mathcal{G}_{\mathbb{R}}$  in all characteristics  $p$ . We begin with material that is independent of  $p$  and then treat  $p = 2$  and  $p > 2$  in turn. Much of this section is adapted from calculations in [2] and [4], to which we sometimes refer for details.

Recall that  $C\mathcal{G}$  is the group of gauge transformations of  $E$  with constant determinant. This fits into a short exact sequence

$$1 \rightarrow S\mathcal{G} \rightarrow C\mathcal{G} \rightarrow \mathbb{C}^* \rightarrow 1$$

where  $S\mathcal{G}$  is the group of gauge transformations with determinant 1. Likewise we have a short exact sequence

$$(3.1) \quad 1 \rightarrow S\mathcal{G}_{\mathbb{R}} \rightarrow C\mathcal{G}_{\mathbb{R}} \rightarrow \mathbb{R}^* \rightarrow 1$$

where  $S\mathcal{G}_{\mathbb{R}}$  and  $C\mathcal{G}_{\mathbb{R}}$  are the subgroups of  $S\mathcal{G}$  and  $C\mathcal{G}$  respectively that commute with  $\tilde{\tau}$ .

From the classification of  $C^\infty$ -Real vector bundles over a real curve in [7], it is always possible to decompose into Real subbundles

$$(3.2) \quad E = E' \oplus L$$

where  $L$  is a Real line bundle. Define a splitting of (3.1) by lifting  $\lambda \in \mathbb{R}^*$  to the real gauge transformation which is trivial on  $E'$  and scalar multiplication by  $\lambda$  on  $L$ . This implies that  $C\mathcal{G}_{\mathbb{R}}$  is isomorphic to a semi-direct product  $\mathbb{R}^* \ltimes S\mathcal{G}_{\mathbb{R}}$ .

Suppose now that  $E$  is endowed with a  $\tilde{\tau}$ -equivariant Hermitian metric and let  $S\mathcal{U}_{\mathbb{R}} \leq S\mathcal{G}_{\mathbb{R}}$  be the subgroup of elements that act unitarily. This inclusion is a homotopy equivalence, because  $S\mathcal{G}_{\mathbb{R}}/S\mathcal{U}_{\mathbb{R}}$  can be identified with the convex space of  $\tau$ -compatible Hermitian metrics, so it induces a homotopy equivalence

$$BS\mathcal{U}_{\mathbb{R}} \cong BS\mathcal{G}_{\mathbb{R}}.$$

For technical reasons to do with compactness, it is preferable to work with  $S\mathcal{U}_{\mathbb{R}}$ .



Let  $X$  denote a compact orientable 2-manifold of genus  $\hat{g}$  with  $n$  boundary components, where  $2\hat{g} + n - 1 = g$ . Consider the pull-back diagram of groups

$$(3.3) \quad \begin{array}{ccc} SU(X, r; \tau_1, \dots, \tau_n) & \longrightarrow & Maps(X, SU_r) \\ \downarrow & & \downarrow \pi \\ \prod_{i=1}^n LSU_r^{\tau_i} & \xrightarrow{\iota} & \prod_{i=1}^n LSU_r \end{array}$$

where  $Maps(X, SU_r)$  is the space of continuous maps from  $X$  to  $SU(r)$  with pointwise multiplication,  $LSU_r$  is the space of continuous maps from  $S^1 = \mathbb{R}/2\pi\mathbb{Z}$  into  $SU_r$ ,  $\pi$  is restriction onto the boundary circles numbered 1 to  $n$ , and  $\iota$  is the product of inclusions of some choice of real loop groups  $LSU_r^{\tau_i} \leq LSU_r$  that will be introduced shortly. Applying the classifying space functor yields a homotopy pull-back diagram

$$(3.4) \quad \begin{array}{ccc} BSU(X, r; \tau_1, \dots, \tau_n) & \longrightarrow & BMaps(X, SU_r) \\ \downarrow & & \downarrow B\pi \\ \prod_{i=1}^n BLSU_r^{\tau_i} & \xrightarrow{B\iota} & \prod_{i=1}^n BLSU_r. \end{array}$$

We must now describe the Real loop groups  $LSU_r^{\tau_i}$ . These are subgroups of  $LSU_r$  and come in three types:

- ( $\alpha$ )  $LSU_r^{\tilde{\tau}_\alpha} = LSO_r$  sitting inside  $LSU_r$  in the standard way,
- ( $\beta$ )  $LSU_r^{\tilde{\tau}_\beta} = L_{-1}SO_r$  is the group of locally orientation preserving gauge transformations of a Möbius bundle  $\mathbb{R}^r \rightarrow M \rightarrow S^1$ . It injects into  $LSU_r$  via an isomorphism  $M \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}^r \times S^1$ .
- ( $\gamma$ )  $LSU_r^{\tilde{\tau}_\gamma} = \{g : S^1 \rightarrow U_r | g(\theta) = \overline{g(\theta + \pi)}\}$  where the bar means entry-wise complex conjugation.

**Lemma 3.1.** *For some choice of  $\tau_1, \dots, \tau_n \in \{\tau_\alpha, \tau_\beta, \tau_\gamma\}$ , there is an isomorphism  $SU_{\mathbb{R}} \cong SU(X, r; \tau_1, \dots, \tau_n)$  that induces a homotopy equivalence*

$$BSG_{\mathbb{R}} \cong BSU_{\mathbb{R}} \cong BSU(X, r; \tau_1, \dots, \tau_n).$$

*There is one real loop group of type ( $\alpha$ ) for each real component of  $\Sigma^\tau$  over which  $E^{\tilde{\tau}}$  is trivial, one of type ( $\beta$ ) for each real component for which  $E^{\tilde{\tau}}$  is nonorientable, and a positive number of type ( $\gamma$ ) if and only if  $\Sigma \setminus \Sigma^\tau$  is connected.*

*Proof.* This proven the same way as ([2] Proposition 6.2) except that  $U(r)$  is replaced by  $SU(r)$ .  $\square$

Our plan is to compute  $H^*(BSG(X, r; \tau_1, \dots, \tau_n))$  using the Eilenberg-Moore spectral sequence (EMSS) associated to (3.4).

**Lemma 3.2.** *Over any coefficient field we have an isomorphism  $H^*(BLSU_r) \cong \wedge(\bar{c}_2, \dots, \bar{c}_r) \otimes S(c_2, \dots, c_r)$ , where the generators have degrees  $|\bar{c}_k| = 2k - 1$  and  $|c_k| = |c_k| = 2k$ .*

*Proof.* Restricting to the basepoint determines a fibration sequence

$$(3.5) \quad SU_r \rightarrow BLSU_r \rightarrow BSU_r,$$

where we have identified  $B\Omega SU_r$  with  $SU_r$ . The inclusion  $LSU_r \hookrightarrow LU_r$  induces a morphism of fibration sequences of (3.5) into

$$(3.6) \quad U_r \rightarrow BLU_r \rightarrow BU_r.$$

It was proven in [2] Proposition 4.3 (stated for  $\mathbb{Z}_2$  coefficients, but the proof is valid in any characteristic) that the Serre spectral sequence of (3.10) collapses yielding a ring isomorphism

$$H^*(LU_r) \cong H^*(U_r) \otimes H^*(BU_r).$$

Because  $SU(r) \subseteq U(r)$  determines a surjection on cohomology, Leray-Hirsch yields a ring isomorphism

$$H^*(BLSU_r) \cong H^*(SU_r) \otimes H^*(BSU_r).$$

□

For the rest of this section we use index sets,  $i \in \{1, \dots, n\}$ ,  $i' \in \{2, \dots, n\}$ ,  $k \in \{2, \dots, r\}$ . We use the notational convention that the appearance of one of these subscripts means to include the full range of that index set. For example  $\wedge(\bar{c}_2, \dots, \bar{c}_r) \otimes S(c_2, \dots, c_r) = \wedge(\bar{c}_k) \otimes S(c_k)$ .

**Lemma 3.3.** *Over any field  $\mathbb{F}$ , we have an isomorphisms*

$$(3.7) \quad H^*\left(\prod_{i=1}^n BLSU_r\right) \cong \bigotimes_{i=1}^n H^*(BLSU(r)) \cong \wedge(\bar{c}_{i,k}) \otimes S(c_{i,k})$$

and

$$(3.8) \quad H^*(BMaps(X, SU_r)) \cong \frac{\wedge(\bar{c}_{i,k})}{(c_{1,k} + \dots + c_{n,k})} \otimes S(c_k) \otimes A.$$

where the generators have degrees  $|\bar{c}_{i,k}| = 2k - 1$  and  $|c_{i,k}| = |c_k| = 2k$  and  $A$  is an exterior algebra with Poincaré series

$$P_t(A) = \prod_{k=2}^r (1 + t^{2k-1})^{2g}.$$

In terms of these generators, the map

$$B\pi^* : H^*\left(\prod_{i=1}^n BLSU_r\right) \rightarrow H^*(BMaps(X, SU_r))$$

is determined by  $B\pi^*(\bar{c}_{i,k}) = \bar{c}_{i,k}$ , and  $B\pi^*(c_{i,k}) = c_k$ .

*Proof.* Equation (3.7) follows from Lemma 3.2 by the Kunneth Theorem.

To prove (3.8), first observe the homotopy equivalence

$$X \sim \vee_g S^1$$

between  $X$  and a wedge of  $g = 2\hat{g} + n - 1$  circles. Thus

$$BMaps(X, SU_r) \cong BMaps(\vee_g S^1, SU_r).$$

Restricting to the basepoint determines a fibration sequence

$$(3.9) \quad SU(r)^g \rightarrow BMaps(X, SU_r) \rightarrow BSU(r).$$

The inclusion  $Maps(X, SU(r)) \hookrightarrow Maps(X, U(r))$  induces a morphism of fibration sequences of (3.9) into

$$(3.10) \quad U(r)^g \rightarrow BMaps(X, U_r) \rightarrow BU(r).$$

It was proven (stated for  $\mathbb{Z}_2$  coefficients, but the proof is valid in any characteristic) in [2] Lemma 4.4 that the Serre spectral sequence of (3.10) collapses yielding a ring isomorphism

$$H^*(BMaps(X, U_r)) \cong H^*(U(r)^g) \otimes H^*(BU(r)).$$

Because  $SU(r) \leq U(r)$  determines a surjection on cohomology, Leray-Hirsch yields a ring isomorphism

$$H^*(BMaps(X, SU_r)) \cong H^*(SU(r)^g) \otimes H^*(BSU(r)).$$

Under the homotopy equivalence between  $X$  and a wedge of circles,  $(n-1)$  of the boundary circles of  $X$  are sent to circles in the wedge, while the sum of the boundary circles is a boundary. The induced map on cohomology follows.  $\square$

The Koszul-Tate complex for the homomorphism  $B\pi^*$  is identified with the bigraded complex  $(K^{*,*}, \delta)$  where

$$(3.11) \quad K^{*,*} := \Gamma(z_k) \otimes \wedge(x_{i',k}) \otimes \wedge(\bar{c}_{i,k}) \otimes S(c_{i,k}) \otimes A$$

with bidegrees and differentials

generator	bi-degree	$\delta$ -derivative
$\bar{c}_{i,k}$	$(0, 2k-1)$	0
$c_{i,k}$	$(0, 2k)$	0
$x_{i',k}$	$(-1, 2k)$	$c_{i',k} - c_{1,k}$
$z_k$	$(-1, 2k-1)$	$\bar{c}_{1,k} + \dots + \bar{c}_{n,k}$

Note in particular that  $K^{*,*}$  is a free extension over

$$R^* := H^*\left(\prod_{i=1}^n BLSU_r\right) \cong \wedge(\bar{c}_{i,k}) \otimes S(c_{i,k})$$

and the cohomology  $H(K^{*,*}, \delta)$  is isomorphic to  $H^*(BMaps(X, SU_r))$  as a graded  $R$ -module, where we understand elements in  $H^d(BMap(X, SU_r))$  to have bi-degree  $(0, d)$ . By homotopy pullback (3.4) gives rise to a Eilenberg-Moore spectral sequence (EMSS), for which  $EM_2^{*,*}$  is isomorphic as a bi-graded algebra to the homology of the complex

$$(3.12) \quad (K^{*,*} \otimes_{R^*} H^*\left(\prod_{i=1}^n BLSU_r^{\tau_i}\right), \delta \otimes_{R^*} 1).$$

### 3.1. Characteristic 2.

3.1.1. *Cohomology of the loop groups over  $\mathbb{Z}_2$ .* It follows from surjectivity into the non-fixed determinant case that the loop groups

$$B\Omega SO_r \rightarrow BL_\sigma SO_r \rightarrow BSO_r.$$

and

$$B\Omega SU_r \rightarrow BLSU_r^\tau \rightarrow BSU_r$$

have Serre Spectral sequences that collapse. Consequently,

**Proposition 3.4.** *We have isomorphisms*

$$H^*(BL_\sigma SO_r; \mathbb{Z}_2) \cong H^*(SO_r; \mathbb{Z}_2) \otimes H^*(BSO_r; \mathbb{Z}_2)$$

as modules over  $H^*(BSO_r; \mathbb{Z}_2)$  and

$$H^*(BLSU_r^\tau; \mathbb{Z}_2) \cong H^*(SU_r; \mathbb{Z}_2) \otimes H^*(BSU_r; \mathbb{Z}_2)$$

as modules over  $H^*(BSU_r; \mathbb{Z}_2)$ .

**Corollary 3.5.**  *$H^*(BL_\sigma SO_r; \mathbb{Z}_2)$  is a free module over  $H^*(BSU_r; \mathbb{Z}_2)$  with Poincaré polynomial*

$$P_t(BSU_r) \prod_{k=2}^r (1 + t^{k-1})(1 + t^k),$$

and  $H^*(BLSU_r^\tau; \mathbb{Z}_2)$  is a free module over  $H^*(BSU_r; \mathbb{Z}_2)$  with Poincaré polynomial

$$P_t(BSU_r) \prod_{k=2}^r (1 + t^{2i-1}).$$

3.1.2. *Cohomology of  $BC\mathcal{G}_\mathbb{R}$  over  $\mathbb{Z}_2$ .*

**Theorem 3.6.** *The inclusion  $C\mathcal{G}_\mathbb{R} \leq \mathcal{G}_\mathbb{R}$  induces a surjection in cohomology*

$$H^*(B\mathcal{G}_\mathbb{R}; \mathbb{Z}_2) \rightarrow H^*(BC\mathcal{G}_\mathbb{R}; \mathbb{Z}_2).$$

The short exact sequence (2.3) gives rise to a fibration sequence

$$(3.13) \quad BC\mathcal{G}_\mathbb{R} \rightarrow B\mathcal{G}_\mathbb{R} \rightarrow \overline{B\mathcal{G}(1)}_\mathbb{R}.$$

We can save some work by using the following lemma. For a formal power series  $p(t) = \sum_{i=0}^\infty a_i t^i$  and  $q(t) = \sum_{i=0}^\infty b_i t^i$ , introduce the partial order  $p(t) \leq q(t)$  if and only if  $a_i \leq b_i$  for all  $i \in \{0, 1, 2, \dots\}$ .

**Lemma 3.7.** *Suppose that  $F \rightarrow E \rightarrow B$  is a Serre fibration such that  $H^*(F)$  and  $H^*(B)$  are finite dimensional in every degree and  $B$  is homotopy equivalent to a connected cell complex such that for every  $d \geq 0$ , the number of  $d$ -cells equals  $\dim(H^d(B))$ . Then*

$$(3.14) \quad P_t(E) \leq P_t(F)P_t(B)$$

with equality if and only if  $H^*(E) \rightarrow H^*(F)$  is surjective.

*Proof.* The  $E_2$  page of the Serre spectral sequence is  $E_2^{p,q} = H^p(B; H^q(F))$  which is the cohomology of a local system. However, using the cellular decomposition on  $B$  we have  $E_1^{p,q} = H^p(B) \otimes H^q(F)$ . It follows that  $\dim E_\infty^{p,q} \leq \dim(H^p(B) \otimes H^q(F))$  which implies (3.14). Equality only occurs if  $E_\infty^{p,q} \cong H^p(B) \otimes H^q(F)$  for all  $p, q$  which implies that  $E_\infty^{0,q} \cong H^q(F)$  so that  $H^*(E) \rightarrow H^*(F)$  is surjective. The converse is simply the Leray-Hirsch Theorem.  $\square$

By Lemma 2.1, the base of (3.13) is homotopy equivalent to  $(S^1)^g$  which admits a cell decomposition satisfying the hypotheses of Lemma 3.7. The Poincaré series of  $B\mathcal{G}_\mathbb{R}$  was worked out in ([2] Theorem 6.1)

$$\begin{aligned} P_t(B\mathcal{G}_\mathbb{R}; \mathbb{Z}_2) &= \frac{1 - t^{2r}}{(1 + t^r)^a} \prod_{k=1}^r \frac{(1 + t^k)^{2a} (1 + t^{2k-1})^{g+1-a}}{(1 - t^{2k})^2} \\ &= \frac{(1 + t)^g}{1 - t} \prod_{k=2}^r \frac{(1 + t^{k-1})^a (1 + t^k)^a (1 + t^{2k-1})^{g+1-a}}{(1 - t^{2k})(1 - t^{2k-2})} \end{aligned}$$

where  $a$  is the number of real circles in  $(\Sigma, \tau)$ . Thus to prove Theorem 3.6 it suffices to show that

$$(3.15) \quad P_t(BC\mathcal{G}_\mathbb{R}) \leq P_t(B\mathcal{G}_\mathbb{R}) / P_t(\overline{B\mathcal{G}(1)}_\mathbb{R}) = \frac{1}{1 - t} \prod_{k=2}^r \frac{(1 + t^{k-1})^a (1 + t^k)^a (1 + t^{2k-1})^{g+1-a}}{(1 - t^{2k})(1 - t^{2k-2})}$$

The short exact sequence (3.1) determines a fibration sequence  $BS\mathcal{G}_\mathbb{R} \rightarrow BC\mathcal{G}_\mathbb{R} \rightarrow B\mathbb{R}^*$  that also satisfies the conditions of Lemma 3.7 so we find that

$$(3.16) \quad P_t(BC\mathcal{G}_\mathbb{R}) \leq P_t(BS\mathcal{G}_\mathbb{R}) P_t(B\mathbb{R}^*) = P_t(BS\mathcal{G}_\mathbb{R}) / (1 - t).$$

Therefore to prove Theorem 3.6 it suffices to prove the following.

**Proposition 3.8.** *The cohomology ring  $H^*(BS\mathcal{G}_\mathbb{R}; \mathbb{Z}_2)$  has Poincaré series*

$$(3.17) \quad P_t(BS\mathcal{G}_\mathbb{R}; \mathbb{Z}_2) = \prod_{k=2}^r \frac{(1 + t^{k-1})^a (1 + t^k)^a (1 + t^{2k-1})^{g+1-a}}{(1 - t^{2k})(1 - t^{2k-2})}$$

where  $a = \pi_0(\Sigma^\tau)$  and  $g$  is the genus of  $\Sigma$ .

*Proof.* We refer the reader to ([2] Appendix A) or McLeary ([12] 7.1) for background on the Eilenberg-Moore spectral sequence.

Identify  $BS\mathcal{G}_\mathbb{R} = BS\mathcal{G}(\hat{g}, n; \tilde{\tau}_1, \dots, \tilde{\tau}_n)$  from the homotopy pull-back diagram (3.4). The associated Eilenberg-Moore spectral sequence  $EM_r^{*,*}$  converges to  $H^*(BS\mathcal{G}_\mathbb{R})$ . The second page  $EM_2^{*,*}$ , equals the cohomology of the differential bi-graded algebra  $(K^{*,*} \otimes_{R^*} M^*, \delta \otimes 1)$  where

- $(K^{*,*}, \delta)$  is the Koszul-Tate complex (3.11),
- $M^* = M^{0,*} := \bigotimes_{i=1}^n H^*(BLSU(r)^{\tilde{\tau}_i})$ , and

$$\bullet R^* = R^{0,*} := \bigotimes_{i=1}^n H^*(BLSU(r)) = \wedge(\bar{c}_{i,k}) \otimes S(c_{i,k}).$$

Applying Lemma 3.3, we have an isomorphism of graded  $R^*$ -modules

$$M^* \cong V \otimes S(c_{i,k})$$

where  $V$  is a graded vector space with Poincaré series

$$P_t(V) = \prod_{k=2}^r (1+t^{k-1})^a (1+t^k)^a (1+t^{2k-1})^{n-a}.$$

We have an isomorphism  $K^{*,*} \otimes_{R^*} M^* \cong \Gamma(z_k) \otimes V \otimes \wedge(x_{i',k}) \otimes S(c_{i,k}) \otimes A$  where  $\delta(V) = \delta(A) = \delta(c_{i,k}) = \delta(z_k) = 0$  and  $\delta(x_{i',k}) = c_{i',k} + c_{1,k}$ . Therefore

$$EM_2^{*,*} = (K^{*,*} \otimes_{R^*} M^*, \delta \otimes 1) \cong \Gamma(z_k) \otimes V \otimes A \otimes S(c_k).$$

Thus  $EM_2^{*,*}$  has Hilbert series with respect to the total grading equal to

$$P_t(EM_2^{*,*}) = \prod_{k=2}^r \frac{(1+t^{k-1})^a (1+t^k)^a (1+t^{2k-1})^{n-a} (1+t^{2k-1})^{2\hat{g}}}{(1-t^{2k})(1-t^{2k-2})},$$

which equals the right hand side of (3.18) because  $g = 2\hat{g} + n - 1$ . It follows then that

$$P_t(BS\mathcal{G}_{\mathbb{R}}; \mathbb{Z}_2) \leq \prod_{k=2}^r \frac{(1+t^{k-1})^a (1+t^k)^a (1+t^{2k-1})^{g+1-a}}{(1-t^{2k})(1-t^{2k-2})}.$$

Since the reverse inequality was already known, the equality (3.18) holds and the spectral sequence collapses at  $EM_2^{*,*}$ . □

Consequently both inequalities (3.15) and (3.16) are equalities, yielding

**Corollary 3.9.** *The cohomology ring  $H^*(BC\mathcal{G}_{\mathbb{R}}; \mathbb{Z}_2)$  has Poincaré series*

$$(3.18) \quad P_t(BC\mathcal{G}_{\mathbb{R}}; \mathbb{Z}_2) = \frac{1}{1-t} \prod_{k=2}^r \frac{(1+t^{k-1})^a (1+t^k)^a (1+t^{2k-1})^{(g+1-a)}}{(1-t^{2k})(1-t^{2k-2})}$$

where  $a = \pi_0(\Sigma^\tau)$  and  $g$  is the genus of  $\Sigma$ .

**3.2. Characteristic  $\neq 2$ .** Throughout this subsection, let  $\mathbb{F}$  denote a field of odd or zero characteristic. Cohomology will always be taken with coefficients  $\mathbb{F}$ . Since  $S\mathcal{G}_{\mathbb{R}} \leq C\mathcal{G}_{\mathbb{R}}$  has index two, there is a natural identification of  $H^*(BC\mathcal{G}_{\mathbb{R}}; \mathbb{F})$  with the invariant ring  $H^*(BS\mathcal{G}_{\mathbb{R}}; \mathbb{F})^{C\mathcal{G}_{\mathbb{R}}/S\mathcal{G}_{\mathbb{R}}}$  which we will exploit in our calculation.

The action of  $C\mathcal{G}_{\mathbb{R}}/S\mathcal{G}_{\mathbb{R}} \cong \mathbb{Z}/2$  on  $H^*(BS\mathcal{G}_{\mathbb{R}}; \mathbb{F})$  is induced by a group automorphism of  $S\mathcal{G}_{\mathbb{R}}$  determined by conjugating by an element  $g \in C\mathcal{G}_{\mathbb{R}} \setminus S\mathcal{G}_{\mathbb{R}}$ . Using the real decomposition  $E = E' \oplus L$  described in (3.2), we may choose  $g$  to be the gauge transformation which acts trivially on  $E'$  and by  $-1$  on  $L$ . This automorphism extends naturally to the diagram (3.3) and therefore it acts on the spectral sequence (3.12). This automorphism restricts to an inner automorphism on  $\text{Maps}(X, U_r)$  and  $\prod_{i=1}^n LU_r$ . By a theorem

of Segal ([16] §3), the induced action of  $C\mathcal{G}_{\mathbb{R}}/S\mathcal{G}_{\mathbb{R}}$  on  $BMaps(X, U_r)$  and on  $\prod_{i=1}^n BLU_r$  is homotopically trivial so in particular  $C\mathcal{G}_{\mathbb{R}}/S\mathcal{G}_{\mathbb{R}}$  acts trivially on  $K^{*,*}$  and  $R^*$ . Therefore the only non-trivial contribution to the  $C\mathcal{G}_{\mathbb{R}}/S\mathcal{G}_{\mathbb{R}}$  action on (3.12) comes from the action on  $H^*(LSU_r^{\tau_i}; \mathbb{F})$  which we investigate next.

### 3.2.1. Cohomology of Real loop groups in odd or zero characteristic.

**Proposition 3.10.** *Let  $r$  be a positive integer and let  $r = 2r' + 1$  or  $r = 2r'$  depending on whether  $r$  is even or odd. We have isomorphism*

$$H^*(BLSO(2r' + 1); \mathbb{F}) \cong \wedge(\bar{p}_1, \dots, \bar{p}_{r'}) \otimes S(p_1, \dots, p_{r'})$$

and

$$H^*(BLSO(2r'); \mathbb{F}) \cong \wedge(\bar{p}_1, \dots, \bar{p}_{r'-1}, \bar{e}_{r'}) \otimes S(p_1, \dots, p_{r'-1}, e_{r'}).$$

with degrees  $|p_k| = 4k$ ,  $|\bar{p}_k| = 4k - 1$ ,  $|e_{r'}| = 2r'$ , and  $|\bar{e}_{r'}| = 2r' - 1$ . In the even rank case denote  $p_{r'} = e_{r'}^2$  and  $\bar{p}_{r'} = 2\bar{e}_{r'}e_{r'}$  for convenience. The inclusion  $\iota : LSO(r) \hookrightarrow LSU(r)$  induces a morphism on cohomology from  $H^*(BLSU(r)) = \wedge(\bar{c}_2, \dots, \bar{c}_r) \otimes S(c_2, \dots, c_r)$  to  $H^*(BLSO(r))$  satisfying

- $\iota^*(\bar{c}_{2k-1}) = \iota^*(c_{2k-1}) = 0$ , for all  $k$
- $\iota^*(c_{2k}) = p_k$  and  $\iota^*(\bar{c}_{2k}) = \bar{p}_k$  for all  $k$ ,

The conjugation action of  $LO(r)/LSO(r) \cong \mathbb{Z}/2$  on  $H^*(BLSO(r); \mathbb{F})$  is trivial on generators  $p_i$  and  $\bar{p}_i$  for all  $i$  and by  $-1$  on  $e_{r'}$  and  $\bar{e}_{r'}$ .

*Proof.* The formulas for  $H^*(BLSO(r); \mathbb{F})$  can be deduced from ([10] Theorem 2) using the well known fact that  $H^*(BSO(r); \mathbb{F})$  is a polynomial ring generated by Pontryagin classes and (if  $r$  is even) the Euler class. Using the identification  $BLSO(r) \cong LBSO(r)$ , we get an evaluation map  $ev : S^1 \times LBSO(r) \rightarrow BSO(r)$ . The generators are defined by  $p_i := \int_{pt} ev^*(p_i)$  and  $\bar{p}_i := \int_{S^1} ev^*(p_i)$  where  $\int$  denotes the slant product with respect to homology class in  $H_*(S^1; \mathbb{F})$  and  $e_i, \bar{e}_i$  and  $c_i, \bar{c}_i$  are defined similarly. The formula for  $i^*$  follows from the well known relationships between Chern classes, Pontryagin classes, and Euler classes described in [14]. We refer to [2] §4 where this construction is laid out in greater detail.  $\square$

**Corollary 3.11.** *The invariant subring of the  $LO(r)/LSO(r) \cong \mathbb{Z}/2$  automorphism described above satisfies*

$$H^*(BLSO(2r'); \mathbb{F})^{\mathbb{Z}/2} \cong H^*(BLSO(2r'+1); \mathbb{F})^{\mathbb{Z}/2} \cong H^*(BLSO(2r'+1); \mathbb{F}) \cong \wedge(\bar{p}_1, \dots, \bar{p}_{r'}) \otimes S(p_1, \dots, p_{r'})$$

with all isomorphism induced by the obvious inclusions. The induced map  $H^*(BLSU_r; \mathbb{F}) \rightarrow H^*(BLSO(r); \mathbb{F})^{\mathbb{Z}/2}$  sends

$$\begin{aligned} \iota^*(\bar{c}_{2k-1}) &= \iota^*(c_{2k-1}) = 0, & \text{for all } k \\ \iota^*(c_{2k}) &= p_k \text{ and } \iota^*(\bar{c}_{2k}) = \bar{p}_k & \text{for all } k. \end{aligned}$$

**Proposition 3.12.** *If  $r$  is odd, then*

$$H^*(BLSU(r)^{\tau\alpha}; \mathbb{F}) \cong H^*(BLSU_r^{\tau\beta}; \mathbb{F}) \cong H^*(BLSU_r^{\tau\gamma}; \mathbb{F}) \cong H^*(BLSO(r); \mathbb{F}).$$

*If  $r$  is even, then*

$$\begin{aligned} H^*(BLSU(r)^{\tau\alpha}; \mathbb{F}) &\cong H^*(BLSO(r); \mathbb{F}). \\ H^*(BLSU(r)^{\tau\beta}; \mathbb{F}) &\cong H^*(BLSO(r-1); \mathbb{F}) \\ H^*(BLSU(r)^{\tau\gamma}; \mathbb{F}) &\cong H^*(BLSO(r); \mathbb{F})^{\mathbb{Z}/2}. \end{aligned}$$

*In all three cases, the homomorphism  $H^*(BLSU_r; \mathbb{F}) \rightarrow H^*(BLSU_r^{\tau}; \mathbb{F})$  agrees with the homomorphisms described in Propositions 3.10 and 3.11 on generators, up to multiplication by a non-zero scalar.*

*Proof.* In case  $\alpha$  we have an equality  $LSU(r)^{\tau\alpha} = LSO(r)$  so there is nothing to prove.

The cases  $\beta$  and  $\gamma$  can be identified with twisted loop groups, and their cohomology has already been calculated in [3] for characteristic greater than  $r$ . The remaining odd primes can be dealt with as follows. We treat only the case  $\gamma$  in detail since  $\beta$  is dealt with similarly.

First note that since  $H^*(BLSU(r)^{\tau\gamma}; \mathbb{Q}) \rightarrow H^*(BLSO(r); \mathbb{Q})^{\mathbb{Z}/2}$  is known to be surjective from [3] and  $H^*(BLSO(r); \mathbb{Z})$  does not contain  $p$  torsion for any odd  $p$ , it follows that

$$(3.19) \quad H^*(BLSU(r)^{\tau\gamma}; \mathbb{F}) \rightarrow H^*(BLSO(r); \mathbb{F})^{\mathbb{Z}/2}$$

is surjective. We have a short exact sequence  $1 \rightarrow \Omega SU(r) \rightarrow LSU(r)^{\tau\gamma} \rightarrow SU(r) \rightarrow 1$  which gives rise to a fibration sequence  $SU(r) \rightarrow BLSU(r)^{\tau\gamma} \rightarrow BSU(r)$ , where we have employed the homotopy equivalence  $B\Omega SU(r) \cong SU(r)$ . The Serre spectral sequence  $(E_k, \delta_k)$ , with  $\delta_k : E_k^{p,q} \rightarrow E_k^{p+k, q-k+1}$  converges to  $BLSU(r)^{\tau\gamma}$  and has  $E_2^{p,q} = H^q(SU(r); \mathbb{F}) \otimes H^p(BSU(r))$  where  $H^*(SU(r); \mathbb{F}) \otimes H^*(BSU(r)) \cong \wedge(\bar{c}_2, \dots, \bar{c}_r) \otimes S(c_2, \dots, c_r)$ . By the surjectivity of (3.19) the even generators  $\bar{c}_{2i}$  survive to  $E_\infty$  for all  $i$ .

We claim that the odd generators  $\bar{c}_{2i+1}$  are all transgressive, meaning that  $\delta_k(\bar{c}_i) = 0$  for  $k < 2i$ . Since  $E_2^{*,*}$  is torsion free, it suffices to prove this for  $\mathbb{F} = \mathbb{Q}$ , when we know that (3.19) is an isomorphism. Since  $\bar{c}_2$  survives to infinity, the only class that can kill  $c_3$  is  $\bar{c}_3$ . Since we know that  $c_3$  is killed (when  $\mathbb{F} = \mathbb{Q}$ ), it follows that  $\bar{c}_3$  is transgressive, so  $\delta_6(\bar{c}_3) = \lambda c_3$  for some non-zero scalar  $\lambda$ , hence  $\delta_k(\bar{c}_3) = 0$  for  $k < 6$ . By induction, this implies that the only class that can kill  $c_5$  is  $\bar{c}_5$  and so on.

Therefore, we know that for all  $i$ ,  $\delta_{4i+2}(\bar{c}_{2i+1}) = \lambda_i c_{2i+1}$  for some nonzero integer  $\lambda_i$ . It remains to show that the  $\lambda_i$  is not divisible by any odd prime  $p$ . If it were, that would mean  $\bar{c}_{2i+1}$  survives to  $E_\infty$ . But this is not true by the following argument. Consider the family of automorphisms of  $LSU(r)^{\tau\gamma} \leq \text{Maps}(S^1, SU(r))$  obtained by rotating the domain circle. Since this is a path connected family, they all act by isotopies on  $BLSU(r)^{\tau\gamma}$  and hence act trivially on cohomology. However, if we rotate by 180 degrees, this has the effect on the fibre of (3.19) of complex conjugating the matrix entry-wise.



In terms of the cohomology ring  $\wedge(\bar{c}_2, \dots, \bar{c}_r)$  this sends  $\bar{c}_{2i} \mapsto \bar{c}_{2i}$  and  $\bar{c}_{2i+1} \mapsto -\bar{c}_{2i+1}$ . It follows that  $\bar{c}_{2i+1}$  is not the restriction of a class in  $H^*(BSU(r)^{\tau_\gamma}; \mathbb{Z}_p)$  for  $p$  odd hence it does not survive to  $E_\infty$ .

The argument for case  $\beta$  is similar, except it is only the primitive  $\bar{e}_r$  of the Euler class that must be shown to be transgressive and the rotation automorphism must also incorporate the twist coming from the Moebius bundle defining  $LSU_r^{\tau_\beta}$ . Lifting a 360 degree rotation of the circle to the Moebius bundle determines an orientation reversal of the fibres and sends  $\bar{e}_r$  to  $-\bar{e}_r$  and the argument goes through as before.  $\square$

### 3.2.2. Cohomology of $BC\mathcal{G}_{\mathbb{R}}$ in odd or zero characteristic.

**Theorem 3.13.** *Let  $\mathbb{F}$  be a field of odd or zero characteristic.*

**Case 1** *If the rank  $r = 2r' + 1$  is odd, then the Poincaré series equals*

$$(3.20) \quad P_t(BC\mathcal{G}_{\mathbb{R}}; \mathbb{F}) = P_t(BS\mathcal{G}_{\mathbb{R}}) = \prod_{k'=1}^{r'} \frac{(1 + t^{4k'-1})^g (1 + t^{4k'+1})^g}{(1 - t^{4k'})^2}$$

*which depends only on the rank  $r$  and degree  $g$ .*

**Case 2** *If the rank  $r = 2r'$  is even, then the Poincaré series factors*

$$P_t(C\mathcal{G}_{\mathbb{R}}; \mathbb{F}) = F_t G_t$$

*where*

$$F_t = \prod_{k''=1}^{r'-1} \frac{(1 + t^{4k''-1})^g (1 + t^{4k''+1})^g}{(1 - t^{4k''})^2}$$

*depends only on the rank  $r$  and the genus  $g$  and  $G_t$  is defined case by case below.*

*Let  $a$  be the number real circles of  $(\Sigma, \tau)$  of which  $b$  are odd and  $c$  are even with respect to  $(E, \tilde{\tau})$ . Then*

- *If  $a = 0$ , then*

$$G_t = \frac{(1 + t^{2r-1})^g}{(1 - t^{2r})}$$

- *If  $a > c \geq 0$  and  $\Sigma \setminus \Sigma^\tau$  is connected then*

$$G_t = \frac{(1 + t^{r-1})^c (1 + t^r)^c + (1 - t^{r-1})^c (1 - t^r)^c}{2} (1 + t^{2r-1})^{g-c-1}$$

- *If  $a = c > 0$  and  $\Sigma \setminus \Sigma^\tau$  is connected then*

$$G_t = \frac{(1 + t^{r-1})^c (1 + t^r)^{c-1} + (1 - t^{r-1})^c (1 - t^r)^{c-1}}{2(1 - t^r)} (1 + t^{2r-1})^{g-c}$$

- *If  $a > c = 0$  and  $\Sigma \setminus \Sigma^\tau$  is disconnected then*

$$G_t = \frac{(1 + t^{2r-1})^g}{1 - t^{2r-2}}$$

- If  $a > c > 0$ ,  $c$  is odd, and  $\Sigma \setminus \Sigma^\tau$  is disconnected then

$$G_t = \frac{(1+t^{r-1})^c(1+t^r)^c + (1-t^{r-1})^c(1-t^r)^c}{2}(1+t^{2r-1})^{g-c-1}$$

- If  $a > c > 0$ ,  $c$  is even, and  $\Sigma \setminus \Sigma^\tau$  is disconnected then

$$G_t = \left( \frac{t^{r-1}(t^{r-1}+t^r)^{c+1}}{(1-t^{2r-2})} + \frac{(1+t^{r-1})^c(1+t^r)^c + (1-t^{r-1})^c(1-t^r)^c}{2} \right) (1+t^{2r-1})^{g-c-1}$$

- If  $a = c > 0$  and  $\Sigma \setminus \Sigma^\tau$  is disconnected then

$$G_t = \frac{(1+t^{r-1})^c(1+t^r)^{c-1} + (1-t^{r-1})^c(1-t^r)^{c-1}}{2(1-t^{2r})}(1+t^{2r-1})^{g-c}$$

*Proof.* Denote  $BS\mathcal{G}_\mathbb{R} = BS\mathcal{G}(\hat{g}, n; \tilde{\tau}_1, \dots, \tilde{\tau}_n)$  from the homotopy pull-back diagram (3.4). The associated Eilenberg-Moore spectral sequence  $EM_r^{*,*}$  converges to  $H^*(BS\mathcal{G}_\mathbb{R}; \mathbb{F})$ . The second page  $EM_2^{*,*}$ , equals the cohomology of the differential bi-graded algebra  $(K^{*,*} \otimes_{R^*} M^*, \delta \otimes 1)$  described in (3.12) where

- $(K^{*,*}, \delta)$  is the Koszul-Tate complex (3.11),
- $M^* = M^{0,*} := \bigotimes_{i=1}^n H^*(BLSU(r)^{\tilde{\tau}_i})$ , and
- $R^* = R^{0,*} := \bigotimes_{i=1}^n H^*(BLSU(r)) = \wedge(\bar{c}_{i,k}) \otimes S(c_{i,k})$ .

**Case 1:  $r = 2r' + 1$  is odd**

In this case  $C\mathcal{G}_\mathbb{R} \cong S\mathcal{G}_\mathbb{R} \times \mathbb{R}^*$ , so  $H^*(BC\mathcal{G}_\mathbb{R}; \mathbb{F}) \cong H^*(BS\mathcal{G}_\mathbb{R} \times B\mathbb{R}^*; \mathbb{F}) \cong H^*(BS\mathcal{G}_\mathbb{R}; \mathbb{F})$ , because  $B\mathbb{R}^* = \mathbb{R}P^\infty$  is  $\mathbb{F}$ -acyclic. So it suffices to compute  $H^*(BS\mathcal{G}_\mathbb{R}; \mathbb{F})$ .

Recall that we have index sets  $i \in \{1, \dots, n\}$ ,  $i' \in \{2, \dots, n\}$ ,  $k \in \{2, \dots, r\}$  and introduce a further index set  $k' \in \{1, \dots, r'\}$ . We have

$$K^{*,*} \otimes_{R^*} M^* = K^{*,*} := \Gamma(z_k) \otimes \wedge(x_{i',k}) \otimes \wedge(\bar{p}_{i,k'}) \otimes S(p_{i,k'}) \otimes A$$

with bidegrees and differentials

generator	bi-degree	$\delta$ -derivative
$\bar{p}_{i,k'}$	$(0, 4k' - 1)$	0
$p_{i,k'}$	$(0, 4k')$	0
$x_{i',2k'}$	$(-1, 4k')$	$p_{i',k'} - p_{1,k'}$
$x_{i',2k'+1}$	$(-1, 4k' + 2)$	0
$z_{2k'}$	$(-1, 4k' - 1)$	$\bar{p}_{1,k'} + \dots + \bar{p}_{n,k'}$
$z_{2k'+1}$	$(-1, 4k' + 1)$	0

Taking cohomology yields

$$(3.21) \quad EM_2^{*,*} = \Gamma(z_{2k'+1}) \otimes \wedge(x_{i',2k'+1}) \otimes \frac{\wedge(\bar{p}_{i,k'})}{(\bar{p}_{i,1} + \dots + \bar{p}_{i,r'})} \otimes S(p_{i,k'}) \otimes A$$

where we abuse notation as usual and denote cohomology classes by representative cocycles.

Over the rational coefficients,  $\Gamma(z_{2k'-1}) \cong S(z_{2k'-1})$  so the bigraded ring  $EM_2^{*,*}$  is generated by elements in the  $-1$  and  $0$  columns, which implies that  $EM_2^{*,*} = EM_\infty^{*,*}$ . By the

universal coefficient theorem, the spectral sequence must collapse for all fields under consideration. Therefore (3.21) is isomorphic to an associated graded ring of  $H^*(BS\mathcal{G}_{\mathbb{R}}; \mathbb{F})$ , yielding (3.20).

**Case 2:  $r = 2r'$  is even**

We suppose that the first  $0 \leq a \leq n$  boundary circles are real circles, and that the first  $0 \leq b \leq a$  have SW class one and the remaining have zero.

We introduce another index set  $k'' \in \{1, \dots, r' - 1\}$ . Applying Proposition 3.12 we have

$$\begin{aligned} K^{*,*} \otimes_{R^*} M^* &= \Gamma(z_k) \otimes \wedge(x_{i',k}) \otimes \wedge(p_{i,k''}) \otimes S(p_{i,k''}) \otimes A \\ &\quad \otimes \wedge(\bar{e}_{b+1}, \dots, \bar{e}_a) \otimes S(e_{b+1}, \dots, e_a) \otimes \wedge(\bar{p}_{a+1,r'}, \dots, \bar{p}_{n,r'}) \otimes S(p_{a+1,r'}, \dots, p_{n,r'}) \end{aligned}$$

with bidegrees and differentials

generator	bi-degree	$\delta$ -derivative
$\bar{p}_{i,k'}$	$(0, 4k' - 1)$	0
$p_{i,k'}$	$(0, 4k')$	0
$e_i$	$(0, r)$	0
$\bar{e}_i$	$(0, r - 1)$	0
$x_{i',2k''}$	$(-1, 4k')$	$p_{i',k''} - p_{1,k''}$
$x_{i',2r'}$	$(-1, 4r')$	$\begin{cases} p_{i',r'} - p_{1,r'} & \text{if } b = 0 \\ p_{i',r'} & \text{if } i' > b > 0 \\ 0 & \text{if } b \geq i' \end{cases}$
$x_{i',2k''+1}$	$(-1, 4k'' + 2)$	0
$z_{2k''}$	$(-1, 4k'' - 1)$	$\bar{p}_{1,k''} + \dots + \bar{p}_{n,k''}$
$z_{2r'}$	$(-1, 4r' - 1)$	$\bar{p}_{b+1,r'} + \dots + \bar{p}_{n,r'}$
$z_{2k''+1}$	$(-1, 4k'' + 1)$	0

where recall we denote  $e_i^2 = p_{i,r'}$  and  $2\bar{e}_i e_i = \bar{p}_{i,r'}$  for  $i \in \{b+1, \dots, a\}$ .

This decomposes as a tensor product of dgas,

$$K^{*,*} \otimes_{R^*} M^* = S^{*,*} \otimes T^{*,*}$$

where

$$\begin{aligned} S^{*,*} &= \Gamma(z_k | k < r) \otimes \wedge(x_{i',k} | k < r) \otimes \wedge(p_{i,k''}) \otimes S(p_{i,k''}) \otimes A \\ T^{*,*} &= \Gamma(z_r) \otimes \wedge(x_{i',r}) \otimes \wedge(\bar{e}_{b+1}, \dots, \bar{e}_a) \otimes S(e_{b+1}, \dots, e_a) \otimes \wedge(\bar{p}_{a+1,r'}, \dots, \bar{p}_{n,r'}) \otimes S(p_{a+1,r'}, \dots, p_{n,r'}) \end{aligned}$$

so we may use the Kunneth formula

$$EM_2^{*,*} = H(K^{*,*} \otimes_{R^*} M^*) = H(S^{*,*}) \otimes H(T^{*,*}).$$

Note that  $S^{*,*}$  is independent of  $a$  or  $b$  with cohomology easily computed

$$H(S^{*,*}) \cong \Gamma(z_{2k''+1}) \otimes \frac{\wedge(\bar{p}_{i,k''})}{(\bar{p}_{1,k''} + \dots + \bar{p}_{n,k''})} \otimes \wedge(x_{i',2k''+1}) \otimes S(p_{k''}) \otimes A.$$

and Poincaré series

$$\begin{aligned} P_t(H(S^{*,*})) &= \left( \prod_{k''=2}^{r'-1} \frac{(1+t^{4k''-1})^{n-1}(1+t^{4k''+1})^{n-1}}{(1-t^{4k''})^2} \right) \prod_{k=2}^{2r'} (1+t^{2k-1})^{2\hat{g}} \\ &= (1+t^{2r-1})^{2\hat{g}} \prod_{k''=2}^{r'-1} \frac{(1+t^{4k''-1})^g(1+t^{4k''+1})^g}{(1-t^{4k''})^2} \end{aligned}$$

Our next task is to calculate the Betti numbers of  $H(T^{*,*})$ . Since we are ultimately interested in  $H(T^{*,*})$  as a graded ring with  $\mathbb{Z}/2$ -action, we will consider  $P_t(H(T^{*,*}))$  with coefficients lying in the ring of characters for  $\mathbb{Z}/2$  where 1 denotes the character of the trivial irrep and  $\chi$  of the non-trivial irrep of  $\mathbb{Z}/2$ .

To calculate the Betti numbers of  $H(T^{*,*})$  we use a filtration of  $T^{*,*}$  and consider the associated trigraded spectral sequence  $E_*^{*,*,*}$  converging to  $H(T^{*,*})$ . Consider the filtration by bigraded dga ideals

$$T^{*,*} = F^0 \supset F^1 \supset \dots \supset F^{a-b+1} = 0$$

where  $F_k := \wedge^{\geq k}(\bar{e}_{b+1}, \dots, \bar{e}_a)T^{*,*}$ . Taking subquotients determines a differential tri-graded algebra  $E_1^{*,*,*}$  such that  $E_1^{*,*,k} = F^k/F^{k+1}$  and  $\delta : E_1^{p,q,k} \rightarrow E_1^{p+1,q,k}$ . If we ignore the third grading, then there is an isomorphism of bigraded algebras  $E_1^{*,*,*} \cong T^{*,*}$ , but it does not respect differentials. For  $\bar{e}_I \in \wedge^k(\bar{e}_{b+1}, \dots, \bar{e}_a)$ , the differential on  $E_1^{*,*,*}$  is determined by the identities

$$\begin{aligned} \delta(\bar{e}_I z_r) &= \bar{e}_I(\bar{p}_{a+1} + \dots + \bar{p}_{n,r'}) \\ \delta(\bar{e}_I x_{i',r}) &= \begin{cases} \bar{e}_I(p_{i',r'} - p_{1,r'}) & \text{if } a = 0 \\ \bar{e}_I p_{i',r'} & \text{if } i' > a > 0 \\ \bar{e}_I e_{i',r'}^2 & \text{if } a \geq i' > b \\ 0 & \text{if } b \geq i' \end{cases} \end{aligned}$$

Define  $E_2^{*,*,*} := H(E_1^{*,*,*}, \delta)$ . We consider three different cases in order of increasing difficulty.

**Case i:**  $a = 0$

In this case  $F_1 = 0$  and the filtration is trivial. We have

$$T^{*,*} = \Gamma(z_r) \otimes \wedge(x_{i',r}) \otimes \wedge(\bar{p}_{1,r'}, \dots, \bar{p}_{n,r'}) \otimes S(p_{1,r'}, \dots, p_{n,r'})$$

and

$$H(T^{*,*}) = \frac{\wedge(\bar{p}_{1,r'}, \dots, \bar{p}_{n,r'})}{(\bar{p}_{1,r'} + \dots + \bar{p}_{n,r'})} \otimes S(p_{r'}).$$

so

$$P_t(H(T^{*,*})) = \frac{(1+t^{2r-1})^{n-1}}{(1-t^{2r})}.$$

**Case ii**  $0 < a < n$  In this case

$$E_1^{*,*,*} = \Gamma(z_r) \otimes \wedge(x_{i',r}) \otimes \wedge(\bar{e}_{b+1}, \dots, \bar{e}_a) \otimes S(e_{b+1}, \dots, e_a) \otimes \wedge(\bar{p}_{a+1,r'}, \dots, \bar{p}_{n,r'}) \otimes S(p_{a+1,r'}, \dots, p_{n,r'}).$$

If  $b > 0$  we get

$$E_2^{*,*,*} := H(E_1^{*,*,*}, \delta) \cong \wedge(x_{2,r}, \dots, x_{b,r}) \otimes \wedge^k(\bar{e}_{b+1}, \dots, \bar{e}_a) \otimes \frac{S(e_{b+1}, \dots, e_a)}{(e_{b+1}^2, \dots, e_a^2)} \otimes \frac{\wedge(\bar{p}_{a+1,r'}, \dots, \bar{p}_{n,r'})}{(\bar{p}_{a+1,r'} + \dots + \bar{p}_{n,r'})}$$

and if  $b = 0$  we get

$$E_2^{*,*,*} = \wedge^k(\bar{e}_1, \dots, \bar{e}_a) \otimes S(e_1) \otimes \frac{S(e_2, \dots, e_a)}{(e_2^2, \dots, e_a^2)} \otimes \frac{\wedge(\bar{p}_{a+1,r'}, \dots, \bar{p}_{n,r'})}{(\bar{p}_{a+1,r'} + \dots + \bar{p}_{n,r'})}.$$

Notice that in both cases, the classes in  $E_2^{*,*,*}$  are represented by cycles in  $T^{*,*}$ . It follows that  $E_2^{*,*,*} = E_\infty^{*,*,*}$  and that we get an isomorphism of bigraded vector spaces  $H(T^{p,q}) = \sum_k E_2^{p,q,k}$  yielding

$$P_t(H(T^{*,*})) = (1 + \chi t^{r-1})^{a-b} (1 + \chi t^r)^{a-b} (1 + t^{2r-1})^{n-a+b-2}$$

if  $b > 0$  and

$$P_t(H(T^{*,*})) = \frac{(1 + \chi t^{r-1})^a (1 + \chi t^r)^{a-1} (1 + t^{2r-1})^{n-a-1}}{1 - t^r}$$

if  $b = 0$ . Furthermore,  $H(T^{*,*})$  is generated as ring by elements lying in columns  $(0, *)$  and  $(1, *)$ .

**Case iii:**  $a = n$

If  $b = n$  then  $T^{*,*} = \Gamma(z_r) \otimes \wedge(x_{i',r})$  and the coboundary map is trivial so  $T^{*,*} = H(T^{*,*})$ .

If  $b \neq n$  then

$$E_1^{*,*,*} = \Gamma(z_r) \otimes \wedge(x_{i',r}) \otimes \wedge(\bar{e}_{b+1}, \dots, \bar{e}_n) \otimes S(e_{b+1}, \dots, e_n).$$

If  $n > b > 0$ , then

$$E_2^{*,*,*} \cong \Gamma(z_r) \otimes \wedge(x_{2,r}, \dots, x_b) \otimes \wedge(\bar{e}_{b+1}, \dots, \bar{e}_n) \otimes \frac{S(e_{b+1}, \dots, e_n)}{(e_{b+1}^2, \dots, e_n^2)}$$

and if  $b = 0$ , then

$$E_2^{*,*,*} \cong \Gamma(z_r) \otimes \wedge(\bar{e}_1, \dots, \bar{e}_n) \otimes S(e_1) \otimes \frac{S(e_2, \dots, e_n)}{(e_2^2, \dots, e_n^2)}.$$

We must now calculate  $E_3^{*,*,*}$ . The boundary map for  $E_2^{*,*,*}$  is determined by  $\delta(z_r) = \bar{p} := \bar{e}_{b+1}e_{b+1} + \dots + \bar{e}_n e_n$ . Observe that  $\bar{p}^2 = 0$ .

Define

$$S := \begin{cases} \wedge(\bar{e}_{b+1}, \dots, \bar{e}_n) \otimes \frac{S(e_{b+1}, \dots, e_n)}{(e_{b+1}^2, \dots, e_n^2)} & \text{if } b > 0 \\ \wedge(\bar{e}_1, \dots, \bar{e}_n) \otimes S(e_1) \otimes \frac{S(e_2, \dots, e_n)}{(e_2^2, \dots, e_n^2)} & \text{if } b = 0 \end{cases}$$

and denote  $\text{Ann}(S) := \{s \in S \mid ps = 0\}$  the annihilator of  $\bar{p}$ . Consider the chain complex

$$\dots \rightarrow^\delta Sz_r^{[3]} \rightarrow^\delta Sz_r^{[2]} \rightarrow^\delta Sz_r \rightarrow^\delta S$$

where the boundary map is  $\delta(sz_r^{[d]}) = s\bar{p}z_r^{[d-1]}$  for any  $s \in S$ . It is clear from this point of view that

$$(3.22) \quad E_3^{*,*,*} \cong \wedge(x_{2,r}, \dots, x_{b,r}) \otimes \left( \frac{S}{\bar{p}S} \oplus \left( \bigoplus_{d=1}^{\infty} H(S)z_r^{[d]} \right) \right)$$

$$(3.23) \quad = \wedge(x_{2,r}, \dots, x_{b,r}) \otimes \left( \frac{S + \text{Ann}(\bar{p}) \otimes \Gamma(z_r)}{\bar{p}S} \right)$$

where  $H(S) := \text{Ann}(\bar{p})/\bar{p}S$ . Since these generators lift to cycles in  $T^{*,*}$  it follows that  $E_3^{*,*,*} = E_{\infty}^{*,*,*}$  and that

$$(3.24) \quad H(T^{*,*}) = \sum_k E_3^{*,*,k}.$$

Furthermore, note that if  $\mathbb{F} = \mathbb{Q}$ , then  $\Gamma(z_r) = S(z_r)$ , so we can choose generators of  $H(T^{*,*})$  lying in columns  $(0, *)$  and  $(-1, *)$ .

**Lemma 3.14.**

$$H(S) = \begin{cases} \mathbb{F}\{\bar{e}_{b+1}, e_{b+1}\} \otimes \dots \otimes \mathbb{F}\{\bar{e}_n, e_n\} & \text{if } b > 0 \\ 0 & \text{if } b = 0. \end{cases}$$

.

*Proof.* The group  $G := \{\pm 1\}^{\{b+1, \dots, n\}} \cong (\mathbb{Z}/2)^{n-b}$  acts by automorphisms on  $S$  where  $g \in G$  acts by

$$g \cdot e_i = g(i)e_i$$

$$g \cdot \bar{e}_i = g(i)\bar{e}_i$$

Since  $G$  stabilizes  $\bar{p}$ , the action restricts to both  $\text{Ann}(\bar{p})$  and  $\bar{p}S$  and thus descends to  $H(S)$ . Let  $g_q \in G$  be the element

$$g_q(i) = \begin{cases} 1 & \text{if } i \neq q \\ -1 & \text{if } i = q \end{cases}$$

and denote  $S^{g_q}$  the subring of  $g_q$  invariants. Let  $\bar{p}_q = \bar{e}_q e_q$ .

Assume that  $n > b > 0$ . Then

$$S^{g_n} = \wedge(\bar{e}_{b+1}, \dots, \bar{e}_{n-1}) \otimes \frac{S(e_{b+1}, \dots, e_{n-1})}{(e_{b+1}^2, \dots, e_{n-1}^2)} \otimes \wedge(\bar{p}_n) \cong \frac{S(e_{b+1}, \dots, e_{n-1})}{(e_{b+1}^2, \dots, e_{n-1}^2)} \otimes \wedge(\bar{p})$$

where in the last step we have changed variables to replace  $\bar{p}_n$  with  $\bar{p} = \bar{e}_{b+1}e_{b+1} + \dots + \frac{1}{2}\bar{p}_n$ . It is clear then that  $\text{Ann}(\bar{p})^{g_n} = (\bar{p}S)^{g_n}$  so  $H(S)^{g_n} = 0$ . Similarly,  $H(S)^{g_q} = 0$  for all  $q \in \{b+1, \dots, n\}$ . It follows that every non-zero element of  $H(S)$  transforms by  $-1$  under  $g_q$  for every  $q \in \{b+1, \dots, n\}$ . The corresponding weight space in  $S$  is

$$S_{(-1, \dots, -1)} = \mathbb{F}\{\bar{e}_{b+1}, e_{b+1}\} \otimes \dots \otimes \mathbb{F}\{\bar{e}_n, e_n\}$$

which is annihilated by  $\bar{p}$ , so we have  $H(S) = S_{(-1, \dots, -1)}$ .

Next assume that  $b = 0$ . Applying the analogous argument, we deduce that  $H(S) = H(S)_{(-1, \dots, -1)}$ . But now

$$S_{(-1, \dots, -1)} = \mathbb{F}\{\bar{e}_1 e_1^{2k}, e_1^{2k+1} | k \geq 0\} \otimes \mathbb{F}\{\bar{e}_2, e_2\} \dots \otimes \mathbb{F}\{\bar{e}_n, e_n\}.$$

Clearly  $\text{Ann}(\bar{p})_{(-1, \dots, -1)} = \text{Ann}(\bar{e}_1 e_1)_{(-1, \dots, -1)} = (\bar{e}_1 e_1)S = \bar{p}S$  so we conclude  $H(S) = 0$ .  $\square$

Next observe that

$$\begin{aligned} P_t(S) &= \frac{1}{t^{2r-1}} P_t(pS) + P_t(\text{Ann}(S)) \\ &= \frac{1 + t^{2r-1}}{t^{2r-1}} P_t(pS) + P_t(H(S)) \end{aligned}$$

so

$$P_t(S/pS) = P_t(S) - P_t(pS) = \frac{P_t(S) - P_t(H(S)t^{2r-1}}{1 + t^{2r-1}}.$$

If  $n > b > 0$ , then combining with (3.22) and (3.24) yields

$$\begin{aligned} P_t(H(T^{*,*})) &= (1 + t^{2r-1})^{b-1} \left( \frac{t^{2r-2}}{1 - t^{2r-2}} P_t(H(S)) + P_t(S/pS) \right) \\ &= (1 + t^{2r-1})^{b-1} \left( \frac{t^{2r-2}(\chi t^{r-1} + \chi t^r)^{n-b}}{1 - t^{2r-2}} + \right. \\ &\quad \left. \frac{(1 + \chi t^{r-1})^{n-b}(1 + \chi t^r)^{n-b} - t^{2r-1}(\chi t^{r-1} + \chi t^r)^{n-b}}{1 + t^{2r-1}} \right) \\ &= (1 + t^{2r-1})^{b-2} \left( \frac{(t^{2r-2} + t^{2r-1})(\chi t^{r-1} + \chi t^r)^{n-b}}{(1 - t^{2r-2})} + (1 + \chi t^{r-1})^{n-b}(1 + \chi t^r)^{n-b} \right) \end{aligned}$$

and if  $b = 0$  we get

$$\begin{aligned} P_t(H(T^{*,*})) &= \frac{P_t(S)}{1 + t^{2r-1}} \\ &= \frac{(1 + \chi t^{r-1})^n (1 + \chi t^r)^{n-1}}{(1 + t^{2r-1})(1 - \chi t^r)} \end{aligned}$$

In all cases we see that if  $\mathbb{F} = \mathbb{Q}$ , then  $EM_2^{*,*} = H(R^{*,*}) \otimes H(T^{*,*})$  is generated by elements in the  $(0, *)$  and  $(-1, *)$  columns, which implies  $EM_2^{*,*} = EM_\infty^{*,*}$ . The case for general  $\mathbb{F}$  in odd characteristic follows by the universal coefficient theorem. This means in particular that

$$P_t(BS\mathcal{G}_{\mathbb{R}}) = P_t(EM_2^{*,*}) = P_t(H(S^{*,*}))P_t(H(T^{*,*})).$$

Finally we must consider the action of  $C\mathcal{G}_{\mathbb{R}}/S\mathcal{G}_{\mathbb{R}} \cong \mathbb{Z}/2$  on  $H^*(BS\mathcal{G}_{\mathbb{R}})$ . Since  $\mathbb{Z}/2$  is semisimple over  $\mathbb{F}$  (terminology) we have an isomorphism  $H^*(BS\mathcal{G}_{\mathbb{R}}) \cong EM_2^{*,*}$  as graded  $\mathbb{Z}/2$ -representations. The action on  $EM_2^{*,*}$  sends  $e_i \rightarrow -e_i$  and  $\bar{e}_i \mapsto -\bar{e}_i$  for all

$i \in \{b+1, \dots, a\}$  and acts trivially on the remaining generators. This action is trivial on  $R^{*,*}$ , so we have

$$(EM_2^{*,*})^{\mathbb{Z}/2} = H^*(S^{*,*}) \otimes H(T^{*,*})^{\mathbb{Z}/2}.$$

and

$$P_t(BC\mathcal{G}_{\mathbb{R}}) = \frac{1}{2}P_t(H(S^{*,*}))\left(P_t^{\chi=1}(H(T^{*,*})) + P_t^{\chi=-1}(H(T^{*,*}))\right).$$

Finally, we define

$$F_t = P_t(H(S^{*,*}))\frac{1}{(1+t^{2r-1})^{2\hat{g}}}$$

$$G_t = \frac{(1+t^{2r-1})^{2\hat{g}}}{2}\left(P_t^{\chi=1}(H(T^{*,*})) + P_t^{\chi=-1}(H(T^{*,*}))\right)$$

□

**Corollary 3.15.** *Let  $(\Sigma, \tau)$  be a real curve with a real circles and let  $\xi$  be a real line bundle for which  $c$  real circles are even and let  $r$  be even. Then the polynomial  $G_t$  appearing in Theorem 3.13 satisfies*

$$G_t = \beta_{2r-2}t^{2r-2} + \beta_{2r-1}t^{2r-1} + O(t^{2r})$$

where

$$\beta_{2r-2} = \begin{cases} \binom{c}{2} & \text{if } c \geq 1 \\ 1 & \text{if } c = 0 \text{ and } a \geq 1 \\ 0 & \text{if } c = a = 0 \end{cases}$$

$$\beta_{2r-1} = \begin{cases} g + c^2 - c - 1 & \text{If } a > c > 0 \\ g - 1 & \text{if } a > c = 0 \text{ and } \Sigma \setminus \Sigma^\tau \text{ is connected} \\ g & \text{if } a > c = 0 \text{ and } \Sigma \setminus \Sigma^\tau \text{ is disconnected} \\ g + c^2 - c & \text{if } a = c > 0 \text{ and } \Sigma \setminus \Sigma^\tau \text{ is connected} \\ g + c^2 - 2c & \text{if } a = c > 0 \text{ and } \Sigma \setminus \Sigma^\tau \text{ is disconnected} \\ g & \text{if } a = c = 0 \end{cases}$$

**3.3. Fundamental groups and the proof of Theorems 1.2 and 1.5.** In this section we compute  $\pi_1(BC\mathcal{G}_{\mathbb{R}}) = \pi_0(C\mathcal{G}_{\mathbb{R}})$  and  $\pi_1(B\mathcal{G}_{\mathbb{R}}) = \pi_0(\mathcal{G}_{\mathbb{R}})$ .

We begin with  $\pi_0(S\mathcal{G}_{\mathbb{R}}) = \pi_0(S\mathcal{U}_{\mathbb{R}})$ . By Lemma 3.1 we have a short exact sequence

$$Maps_*(X/\partial X, SU(r)) \rightarrow S\mathcal{U}_{\mathbb{R}} \rightarrow \prod_{i=1}^n LSU(r)^{\tau_i}$$

Observe that  $X/\partial X$  is a 2-dimensional cell complex and  $SU(r)$  is 2-connected, so  $Maps_*(X/\partial X, SU(r))$  is path connected. It follows that

$$\pi_0(S\mathcal{G}_{\mathbb{R}}) \cong \pi_0(S\mathcal{U}_{\mathbb{R}}) \cong \prod_{i=1}^n \pi_0(LSU(r)^{\tau_i}).$$

For real loop groups of type a and b we have a fibration sequence

$$\Omega SO(r) \rightarrow LSU(r)^{\tau_i} \rightarrow SO(r).$$



Since  $SO(r)$  is connected, we see  $\pi_0(LSU(r))$  is the cokernel of a homomorphism  $\pi_1(SO(r)) \rightarrow \pi_0(\Omega SO(r)) = \pi_1(SO(r))$ . For  $r \geq 3$  this is the cokernel of a map  $\mathbb{Z}/2 \rightarrow \mathbb{Z}/2$  which must be  $\mathbb{Z}/2$  since  $H^1(BLSO(r); \mathbb{F}/2) = \mathbb{Z}_2$ . For  $r = 2$ , we get the cokernel of a map from  $\pi_1(SO(2)) \cong \mathbb{Z}$  to itself which the reader can check gives  $\mathbb{Z}$  for type a and  $\mathbb{Z}/2$  for type b.

For type c we have  $\Omega SU(r) \rightarrow LSU(r)^{\tau_c} \rightarrow SU(r)$  which implies  $\pi_0(LSU(r)) = 1$ . Therefore

**Proposition 3.16.** *Suppose  $(\Sigma, \tau)$  is a real curve with  $a$  real circles and  $\xi$  is a real line bundle for which  $b$  circles are odd. We have an isomorphism*

$$\pi_0(S\mathcal{G}_{\mathbb{R}}) = \pi_1(BS\mathcal{G}_{\mathbb{R}}) \cong \begin{cases} (\mathbb{Z}/2)^a & \text{if } r \geq 3 \\ (\mathbb{Z}/2)^b \times (\mathbb{Z})^{a-b} & \text{if } r = 2. \end{cases}$$

**Proposition 3.17.** *We have an isomorphism*

$$\pi_0(C\mathcal{G}_{\mathbb{R}}) \cong \mathbb{Z}/2 \ltimes \pi_0(S\mathcal{G}_{\mathbb{R}})$$

where  $\mathbb{Z}/2$  acts on  $\pi_0(S\mathcal{G}_{\mathbb{R}})$  diagonally: trivially on the  $\mathbb{Z}/2$  factors and by  $-1$  on the  $\mathbb{Z}$  factors.

*Proof of 3.17.* Since (3.1) splits we know  $\pi_0(C\mathcal{G}_{\mathbb{R}}) \cong \mathbb{Z}/2 \ltimes \pi_0(S\mathcal{G}_{\mathbb{R}})$  is a semi-direct product. In terms of the isomorphism in Proposition 3.16,  $\mathbb{Z}/2$  acts by conjugating each  $LU(r)^{\tau_i}$  by a constant real matrix with negative determinant, producing an automorphism of  $\pi_0(LSU_r^{\tau_i})$ . Clearly the automorphism is trivial whenever  $\pi_0(LSU_r^{\tau_i}) \cong \mathbb{Z}/2$ . The one remaining case is when  $LSU_r^{\tau_i} = LSO_2$  in which case  $\pi_0(LSO_2) = \pi_0(\Omega SO_2) \cong \mathbb{Z}$  and the involution acts by  $-1$ .  $\square$

*Proof of Theorem 1.5.* According to Corollary 2.4, we have an isomorphism  $\pi_1(B\overline{C}\mathcal{G}_{\mathbb{R}}) \cong \pi_1(M(2, \xi)^{\tau})$  and according to (2.4) we have  $\pi_1(BC\mathcal{G}_{\mathbb{R}}) \cong \pi_1(B\overline{C}\mathcal{G}_{\mathbb{R}}) \times \mathbb{Z}/2$ . The abelianization of  $\pi_1(BC\mathcal{G}_{\mathbb{R}})$  is  $(\mathbb{Z}/2)^{a+1}$  so  $H_1(M(2, \xi)^{\tau}) \cong (\mathbb{Z}/2)^a$ .  $\square$

*Proof of Theorem 1.2.* Assume that  $H(M(r, \xi_1)^{\tau_1}; \mathbb{Z}) \cong H(M(r, \xi_2)^{\tau_2}; \mathbb{Z})$ . Since  $M(r, \xi_i)^{\tau_i}$  is a closed manifold of dimension  $(r^2)(2g_i - 2)$  it follows that  $g_1 = g_2$ . The equality  $a_1 = a_2$  follows from Theorem 1.5. Denote  $g := g_1 = g_2$  and  $a := a_1 = a_2$ .

Assume further that  $r$  is even and either  $r = 2$  and  $g_1 \geq 5$  or  $r \geq 4$  and  $g_1 \geq 3$ . Recall that by assumption  $\gcd(r, d_i) = 1$ , so  $d_i$  is odd. By (1.2)  $a - c_i$  must also be odd, so in particular  $a > c_i$  and  $c_1 - c_2$  is even. By Corollary 2.4 we have an isomorphism  $H^k(M(r, \xi)^{\tau}; \mathbb{F}) \cong H^k(B\overline{C}\mathcal{G}_{\mathbb{R}}; \mathbb{F})$  for all  $k \leq g(r - 1) - 2 \leq 2r - 1$ . By the universal coefficient theorem and (2.4) we have isomorphisms

$$H^k(M(r, \xi)^{\tau}; \mathbb{F}) \cong H^k(B\overline{C}\mathcal{G}_{\mathbb{R}}; \mathbb{F}) \cong H^k(BC\mathcal{G}_{\mathbb{R}}; \mathbb{F})$$

where  $\mathbb{F}$  has characteristic  $\neq 2$  and  $k \leq 2r - 1$ . Assume without loss of generality that  $c_1 \geq c_2$ . From the formula for  $\beta_{2r-2}$  in Corollary 3.15, it follows that either  $c_1 = c_2$ , or  $c_1 \in \{1, 2\}$  and  $c_2 = 0$ . The case  $c_1 = 1$  can be dismissed because  $c_1 - c_2$  must be even.

If  $c_1 = 2$  then  $\beta_{2r-1} = g + 1$  and if  $c_2 = 0$  then  $\beta_{2r-1} = g$  or  $g - 1$  so this case also leads to a contradiction.

Assume further that  $c = c_1 = c_2$  is even and  $g \geq 6$ . By the coprime condition and (2.1), the number of odd circles must be odd so  $a$  must also be odd. For fixed odd  $g$  and odd  $a$  there is only one possible topological type of  $(\Sigma, \tau)$  so the result holds. For fixed even  $g$  and odd  $a$  there are two topological types for  $(\Sigma, \tau)$  distinguished by whether  $\Sigma \setminus \Sigma^\tau$  is connected or disconnected. By Theorem 3.13, if  $c = 0$  then the corresponding  $BC\mathcal{G}_{\mathbb{R}}$  have different Betti numbers in degree  $2r - 2$  while if  $c > 0$  they have different Betti numbers in degree  $4r - 4$ . Since  $2r - 2 \leq 4r - 4 \leq g(r - 1) - 2$ , it follows that the moduli spaces have different Betti numbers in degree  $4r - 4$ .  $\square$

#### 4. EQUIVARIANT PERFECTION AND THE PROOF OF THEOREM 1.1

The goal of this section is to prove the following theorem.

**Theorem 4.1.** *The real-Harder Narsimhan stratification,*

$$(4.1) \quad \mathcal{C}^{\tilde{\tau}} = \cup_{\mu} C_{\mu}^{\tilde{\tau}}$$

*is  $C\mathcal{G}_{\mathbb{R}}$ -equivariantly perfect with respect to  $\mathbb{Z}_2$ -coefficients. Consequently the induced map  $H_{C\mathcal{G}_{\mathbb{R}}}(\mathcal{C}^{\tilde{\tau}}; \mathbb{Z}_2) \rightarrow H_{C\mathcal{G}_{\mathbb{R}}}(\mathcal{C}_{ss}^{\tilde{\tau}}; \mathbb{Z}_2)$  is surjective.*

The analogous result with  $C\mathcal{G}_{\mathbb{R}}$  replaced with  $\mathcal{G}_{\mathbb{R}}$  was proven in [2]. That proof boils down to showing that the equivariant Euler classes of the normal bundles of each stratum  $C_{\mu}^{\tilde{\tau}}$  is not a zero divisor in the cohomology ring  $H^*(C_{\mu}^{\tilde{\tau}}; \mathbb{Z}_2)$ . This was accomplished using the following version of the Atiyah-Bott Lemma (Lemma 3.1 from [2]):

**Lemma 4.2.** *Let  $G$  be a compact connected Lie group with  $H^*(G; \mathbb{Z})$  torsion free. Let  $X$  be a  $G$ -space of finite type and let  $E \rightarrow X$  be a  $G$ -equivariant  $\mathbb{R}^n$ -vector bundle. Suppose that there exists  $\epsilon \in G$  such that*

- $\epsilon^2$  is the identity in  $G$
- $\epsilon$  acts trivially on  $X$
- $\epsilon$  acts by scalar multiplication by  $-1$  on  $E$ .

*Then the equivariant Euler class  $Eul_G(E)$  is not a zero divisor in  $H_G^*(X; \mathbb{Z}_2)$ .*

Unfortunately, the required element  $\epsilon \in \mathcal{G}_{\mathbb{R}}$  does not lie in  $C\mathcal{G}_{\mathbb{R}}$ . For that reason we must replace  $C\mathcal{G}_{\mathbb{R}}$  with a larger group containing  $\epsilon$ . Recall (2.3) that  $C\mathcal{G}_{\mathbb{R}}$  is equal to the kernel of the natural homomorphism  $\mathcal{G}_{\mathbb{R}} \rightarrow \overline{\mathcal{G}(1)}_{\mathbb{R}}$ . Define  $\widetilde{C\mathcal{G}}_{\mathbb{R}}$  by the short exact sequence

$$1 \rightarrow \widetilde{C\mathcal{G}}_{\mathbb{R}} \rightarrow \mathcal{G}_{\mathbb{R}} \rightarrow \pi_0(\overline{\mathcal{G}(1)}_{\mathbb{R}}) \rightarrow 1.$$

**Proposition 4.3.** *The inclusion  $C\mathcal{G}_{\mathbb{R}} \hookrightarrow \widetilde{C\mathcal{G}}_{\mathbb{R}}$  is a weak homotopy equivalence. Consequently, (4.1) is  $C\mathcal{G}_{\mathbb{R}}$ -equivariantly perfect if and only if it is  $\widetilde{C\mathcal{G}}_{\mathbb{R}}$ -equivariantly perfect.*

*Proof.* It is clear from the definition that the coset space  $\widetilde{C\mathcal{G}}_{\mathbb{R}}/C\mathcal{G}_{\mathbb{R}}$  is homeomorphic to the identity component of  $\overline{\mathcal{G}(1)}_{\mathbb{R}}$ , which was proven to be contractible in Lemma 2.1.  $\square$

**Lemma 4.4.** *For every splitting  $(E, \tilde{\tau}) = (D_1, \tilde{\tau}_1) \oplus \dots \oplus (D_k, \tilde{\tau}_k)$  into  $C^\infty$ -Real bundles we have a surjection*

$$\pi_0(\mathcal{G}(D_1)_{\mathbb{R}}) \times \dots \times \pi_0(\mathcal{G}(D_k)_{\mathbb{R}}) \rightarrow \pi_0(\mathcal{G}(E)_{\mathbb{R}}).$$

*Proof.* It suffices to show that the restricted map  $i : \pi_0(\mathcal{G}(D_1)_{\mathbb{R}}) \rightarrow \pi_0(\mathcal{G}(E)_{\mathbb{R}})$  is surjective. Consider the short exact sequence (2.3). Since  $\overline{\mathcal{G}(1)}_{\mathbb{R}}$  is a  $K(\mathbb{Z}^g, 1)$ , we have a diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \pi_0(C\mathcal{G}(D_1)_{\mathbb{R}}) & \longrightarrow & \pi_0(\mathcal{G}(D_1)_{\mathbb{R}}) & \longrightarrow & \pi_0(\overline{\mathcal{G}(1)}_{\mathbb{R}}) \longrightarrow 1 \\ & & \downarrow i' & & \downarrow i & & \downarrow = \\ 1 & \longrightarrow & \pi_0(C\mathcal{G}(E)_{\mathbb{R}}) & \longrightarrow & \pi_0(\mathcal{G}(E)_{\mathbb{R}}) & \longrightarrow & \pi_0(\overline{\mathcal{G}(1)}_{\mathbb{R}}) \longrightarrow 1 \end{array}$$

The surjectivity of  $i'$  is evident from the description in Proposition 3.17. The surjectivity of  $i$  follows.  $\square$

The following lemma is necessary for the induction step in the calculation of Betti numbers.

**Lemma 4.5.** *Let  $(D_1, \tilde{\tau}_1) \oplus \dots \oplus (D_k, \tilde{\tau}_k) = (E, \tilde{\tau}) = (D'_1, \tilde{\tau}'_1) \oplus \dots \oplus (D'_k, \tilde{\tau}'_k)$  be two different decompositions of  $E$  into  $C^\infty$ -Real subbundles, such that  $(D_i, \tilde{\tau}_i) \cong (D'_i, \tilde{\tau}'_i)$  for all  $i$ . Then there exists  $g \in \widetilde{C\mathcal{G}(E)}_{\mathbb{R}}$  such that  $g(D_i) = D'_i$  for all  $i$ .*

*Proof.* It is clear that simply by summing together these isomorphisms  $D_i \cong D'_i$  that we can find a gauge transformation  $g \in \mathcal{G}(E)_{\mathbb{R}}$  satisfying  $g(D_i) = D'_i$ . The only question is whether we can choose  $g \in \widetilde{C\mathcal{G}}_{\mathbb{R}}$ . But by Lemma 4.4, we can compose  $g$  by an element of  $h \in \mathcal{G}(D'_1)_{\mathbb{R}} \times \dots \times \mathcal{G}(D'_k)_{\mathbb{R}}$  so that  $hg$  lies in the identity component of  $\mathcal{G}_{\mathbb{R}}$  hence must also lie in  $\widetilde{C\mathcal{G}}_{\mathbb{R}}$ .  $\square$

*Proof of Theorem 4.1.* By Lemma 4.5,  $C\mathcal{G}_{\mathbb{R}}$  acts transitively on the set of decompositions  $(D_1, \tilde{\tau}_1) \oplus \dots \oplus (D_k, \tilde{\tau}_k)$  of a given topological type. It follows that there is a homotopy equivalence of homotopy quotients

$$(4.2) \quad (C_{\mu}^{\tilde{\tau}})_{h\widetilde{C\mathcal{G}}_{\mathbb{R}}^{\tilde{\tau}}} \cong (C_{ss}^{\tau_1} \times \dots \times C_{ss}^{\tau_k})_{h\widetilde{C\mathcal{G}}(D_1, \dots, D_k)_{\mathbb{R}}}$$

where

$$\widetilde{C\mathcal{G}}(D_1, \dots, D_k)_{\mathbb{R}} = \mathcal{G}(D_1)_{\mathbb{R}} \times \dots \times \mathcal{G}(D_k)_{\mathbb{R}} \cap \widetilde{C\mathcal{G}}_{\mathbb{R}}.$$

Choose a basepoint  $p \in \Sigma$  that is not fixed by  $\tau$ . Then restricting gauge transformations to the fiber over  $p$  determines a short exact sequence

$$1 \rightarrow \widetilde{C\mathcal{G}}_{bas}(D_1, \dots, D_k)_{\mathbb{R}} \rightarrow \widetilde{C\mathcal{G}}(D_1, \dots, D_k)_{\mathbb{R}} \rightarrow \prod_{i=1}^k GL_{r_i}(\mathbb{C}) \rightarrow 1$$

By forming the homotopy quotient in stages we get an isomorphism

$$H_{\widetilde{C\mathcal{G}}_{\mathbb{R}}}^*(C_{\mu}^{\tilde{\tau}}; \mathbb{Z}_2) \cong H_U^*((\mathcal{C}_{ss}^{\tau_1} \times \dots \times \mathcal{C}_{ss}^{\tau_k})_{h\widetilde{C\mathcal{G}}_{bas}(D_1, \dots, D_k)_{\mathbb{R}}}; \mathbb{Z}_2)$$

where  $U = U(r_1) \times \dots \times U(r_k)$  is the maximal compact subgroup of  $\prod_{i=1}^k GL_{r_i}(\mathbb{C})$ . As explained in ([2] (2.13)), the normal bundle decomposes into a direct sum of subbundles  $N = \bigoplus_{i < j} N_{i,j}$  where the fibre  $(N_{i,j})_{(\bar{\partial}_1, \dots, \bar{\partial}_k)} \cong H^1(D_i^* \otimes D_j, \bar{\partial}_1^* \otimes Id_{D_j} + Id_{D_i^*} \otimes \bar{\partial}_2^*)^{\tau_i^* \otimes \tau_j}$ . The element  $(Id_{r_1}, \dots, -Id_{r_i}, \dots, Id_{r_k}) \in U$  acts by  $-1$  on the summand  $N_{i,j}$  and trivially on the base so by Lemma 4.2, the equivariant Euler class  $Eul_U(N) = \prod_{i < j} Eul_U(N_{i,j})$  is not a zero divisor in  $H_{\widetilde{C\mathcal{G}}_{\mathbb{R}}}^*(C_{\mu}^{\tilde{\tau}}; \mathbb{Z}_2)$ .  $\square$

*Proof of Theorem 1.1.* Combining Theorem 4.1 and Theorem 3.6 implies that the composed map  $H^*(B\mathcal{G}_{\mathbb{R}}; \mathbb{Z}_2) \rightarrow H^*((\mathcal{C}_{ss})_{hC\mathcal{G}_{\mathbb{R}}}; \mathbb{Z}_2)$  is surjective. From (2.4) and Corollary 2.3, it follows that  $H^*(B\overline{C\mathcal{G}}_{\mathbb{R}}; \mathbb{Z}_2) \rightarrow H^*(M(r, \xi)^{\tau}; \mathbb{Z}_2)$  is also surjective. From (2.5)  $H^*(M(r, d)_w^{\tau}; \mathbb{Z}_2) \rightarrow H^*(M(r, \xi)^{\tau}; \mathbb{Z}_2)$  is also surjective, so the result follows by the Leray-Hirsch Theorem.  $\square$

## 5. PROOF OF THEOREM 1.3

The following is adapted mutatis-mutandis from [4]. See §4 of that paper for a more detailed proof. The idea is simply that the normal bundles of the unstable strata are non-orientable, which implies that their Thom spaces must be acyclic, so they contribute nothing to the Morse complex.

**Proposition 5.1.** *Let  $(E, \tilde{\tau})$  be a Real bundle of rank 2 and let  $\mathbb{F}$  be a field of odd characteristic. Suppose that  $g \equiv d \pmod{2}$ . Then there is an isomorphism*

$$H^*(BC\mathcal{G}_{\mathbb{R}}; \mathbb{F}) \cong H_{C\mathcal{G}_{\mathbb{R}}}^*(\mathcal{C}_{ss}^{\tilde{\tau}}; \mathbb{F}).$$

*Proof.* The action of  $C\mathcal{G}_{\mathbb{R}}$  preserves the real Harder-Narasimhan stratification  $\mathcal{C}^{\tilde{\tau}} = \bigcup_{\mu} \mathcal{C}_{\mu}^{\tilde{\tau}}$ , and determines a stratification

$$BC\mathcal{G}_{\mathbb{R}} \cong (\mathcal{C}^{\tilde{\tau}})_{hC\mathcal{G}_{\mathbb{R}}} = \bigcup_{\mu} (\mathcal{C}_{\mu}^{\tilde{\tau}})_{hC\mathcal{G}_{\mathbb{R}}} = (\mathcal{C}_{ss}^{\tilde{\tau}})_{hC\mathcal{G}_{\mathbb{R}}} \cup \left( \bigcup_{\mu \neq ss} (\mathcal{C}_{\mu}^{\tilde{\tau}})_{hC\mathcal{G}_{\mathbb{R}}} \right)$$

Since  $E$  has rank 2 the higher strata correspond to  $C^{\infty}$ -decompositions into Real line bundles  $E = L_1 \oplus L_2$ . The corresponding strata have the form ((4.2) up to homotopy)

$$(\mathcal{C}_{\mu}^{\tilde{\tau}})_{hC\mathcal{G}_{\mathbb{R}}} \cong (\mathcal{C}(L_1)_{ss}^{\tilde{\tau}_1} \times \mathcal{C}(L_2)^{\tilde{\tau}_2})_{hC\mathcal{G}(L_1, L_2)_{\mathbb{R}}}.$$

where  $C\mathcal{G}(L_1, L_2)_{\mathbb{R}} := (\mathcal{G}(L_1)_{\mathbb{R}} \times \mathcal{G}(L_2)_{\mathbb{R}}) \cap C\mathcal{G}_{\mathbb{R}}$ . For line bundles,  $\mathcal{C}(L_i)_{ss}^{\tilde{\tau}_i} = \mathcal{C}(L_i)^{\tilde{\tau}_i}$  is contractible and we have an isomorphism  $\mathcal{G}(1)_{\mathbb{R}} \times \mathbb{R}^* \cong C\mathcal{G}(L_1, L_2)_{\mathbb{R}}$  defined by  $(g, \lambda) \mapsto g \oplus \lambda g^{-1}$  so by Lemma 2.1 and the fact that  $\mathcal{G}(1)_{\mathbb{R}} \cong \overline{\mathcal{G}_{\mathbb{R}}} \times \mathbb{R}^*$  we have

$$(\mathcal{C}_{\mu}^{\tilde{\tau}})_{hC\mathcal{G}_{\mathbb{R}}} \cong B\mathcal{G}(1)_{\mathbb{R}} \cong (S^1)^g \times (\mathbb{R}P^{\infty})^2 = K(Z^{2g} \times (\mathbb{Z}/2)^2, 1).$$

It follows that for every non-trivial 2-fold covering map over  $(\mathcal{C}_{\mu}^{\tilde{\tau}})_{hC\mathcal{G}_{\mathbb{R}}}$  induces an isomorphism in  $\mathbb{F}$ -cohomology or equivalently, that every non-trivial rank one  $\mathbb{F}$ -local system over  $(\mathcal{C}_{\mu}^{\tilde{\tau}})_{hC\mathcal{G}_{\mathbb{R}}}$  is acyclic. This implies that the Thom space of any non-orientable vector bundle over  $(\mathcal{C}_{\mu}^{\tilde{\tau}})_{hC\mathcal{G}_{\mathbb{R}}}$  must be  $\mathbb{F}$ -acyclic. The normal bundle of  $\mathcal{C}_{\mu}^{\tilde{\tau}}$  has odd rank and the constant scalar  $1 \oplus -1 \in C\mathcal{G}(L_1, L_2)_{\mathbb{R}}$  acts by scalar multiplication by  $-1$  on  $N$ , so the restriction of  $N_{hC\mathcal{G}_{\mathbb{R}}}$  to the second factor of  $\mathbb{R}P^{\infty}$  is non-orientable, thus the Thom space of  $N_{hC\mathcal{G}_{\mathbb{R}}}$  is  $\mathbb{F}$ -acyclic. Since this is true for every stratum except the semi-stable stratum, the result is proven.  $\square$

*Proof of Theorem 1.3.* Since  $\mathbb{R}P^{\infty}$  is  $\mathbb{F}$ -acyclic, by Corollary 2.3 we have

$$H^*(BC\mathcal{G}_{\mathbb{R}}; \mathbb{F}) \cong H_{C\mathcal{G}_{\mathbb{R}}}^*(\mathcal{C}_{ss}^{\tilde{\tau}}; \mathbb{F}) \cong H^*(M(2, \xi) \times \mathbb{R}P^{\infty}; \mathbb{F}) \cong H^*(M(2, \xi); \mathbb{F}).$$

The Poincaré series can be read off from Theorem 3.13.  $\square$

## 6. BETTI NUMBERS

We are now able to compute some Poincaré polynomials. For rank  $r = 1$ ,  $M(1, \xi) = M(1, \xi)^{\tau}$  is just a point. The first interesting case is when  $r = 2$ .

**Proposition 6.1.** *Let  $\Sigma$  be a genus  $g$  real curve with  $a > 0$  real path components and let  $\xi$  be a Real line bundle over  $(\Sigma, \tau)$  of odd degree. The moduli space  $M(2, d, \tau)$  of real bundles of rank two, odd degree  $d$  and fixed topological type has Poincaré series*

$$(6.1) \quad P_t(M(2, \xi)^{\tau}; \mathbb{Z}_2) = \frac{(1+t)^{a-1}(1+t^2)^{a-1}(1+t^3)^{g-\alpha} - 2^{a-1}t^g(1+t)^g}{(1-t)(1-t^2)}$$

where  $a = \pi_0(\Sigma^{\tau})$ .

For example, for a real curve of genus  $g = 2$ , respectively  $a = 1, 2, 3$  real circles,  $P_t(M(2, \xi)^{\tau}; \mathbb{Z}_2)$  equals

$$\begin{aligned} t^3 + t^2 + t + 1 \\ t^3 + 2t^2 + 2t + 1 \\ t^3 + 3t^2 + 3t + 1 \end{aligned}$$

For a real curve of genus  $g = 3$ ,  $a = 1, 2, 3, 4$  real circles,  $P_t(M(2, \xi)^{\tau}; \mathbb{Z}_2)$  equals

$$\begin{aligned} t^6 + t^5 + 2t^4 + 4t^3 + 2t^2 + t + 1 \\ t^6 + 2t^5 + 4t^4 + 6t^3 + 4t^2 + 2t + 1 \\ t^6 + 3t^5 + 7t^4 + 10t^3 + 7t^2 + 3t + 1 \end{aligned}$$

$$t^6 + 4t^5 + 11t^4 + 16t^3 + 11t^2 + 4t + 1$$

This can be compared  $P_t(M(2, \xi)^\tau; \mathbb{Z}_p)$ , for  $p$  odd, for a real curve of genus  $g = 3$  with respectively  $c = 0, 1, 2, 3$  even circles:

$$t^6 + 2t^3 + 1$$

$$t^6 + 2t^3 + 1$$

$$t^6 + t^4 + 4t^3 + t^2 + 1$$

$$t^6 + 3t^4 + 8t^3 + 3t^2 + 1$$

For a real curve of genus  $g = 4$ ,  $a = 1, 2, 3, 4, 5$  real circles,  $P_t(M(2, \xi)^\tau; \mathbb{Z}_2)$  equals

$$t^9 + t^8 + 2t^7 + 6t^6 + 6t^5 + 6t^4 + 6t^3 + 2t^2 + t + 1$$

$$t^9 + 2t^8 + 4t^7 + 9t^6 + 12t^5 + 12t^4 + 9t^3 + 4t^2 + 2t + 1$$

$$t^9 + 3t^8 + 7t^7 + 15t^6 + 22t^5 + 22t^4 + 15t^3 + 7t^2 + 3t + 1$$

$$t^9 + 4t^8 + 11t^7 + 25t^6 + 39t^5 + 39t^4 + 25t^3 + 11t^2 + 4t + 1$$

$$t^9 + 5t^8 + 16t^7 + 40t^6 + 66t^5 + 66t^4 + 40t^3 + 16t^2 + 5t + 1$$

**Proposition 6.2.** *Let  $\Sigma$  be a genus  $g$  real curve with  $a > 0$  real path components and let  $\xi$  be a Real line bundle over  $(\Sigma, \tau)$  of degree not divisible by three. The moduli space  $M(2, \xi)^\tau$  has  $\mathbb{Z}_2$  Poincaré series*

$$\begin{aligned} P_t(M(3, \xi)^\tau; \mathbb{Z}_2) = & \frac{(1+t)^b(1+t^2)^{2b}(1+t^3)^g(1+t^5)^{g-b}}{(1-t)(1-t^2)^2(1-t^3)} \\ & - 2^b \frac{t^{2g}(1+t)^{g+b}(1+t^2)^b(1+t^3)^{g-b}}{t(1-t)^3(1-t^3)} \\ & + 4^b \frac{t^{3g}(1+t)^{2g}(1+t^2+t^4)}{t(1-t)^2(1-t^2)(1-t^6)} \end{aligned}$$

In genus  $g = 2$  and  $a = 1, 2, 3$ ,  $P_t(M(3, 1, \tau))$  equals

$$t^8 + t^7 + 3t^6 + 5t^5 + 4t^4 + 5t^3 + 3t^2 + t + 1$$

$$t^8 + 2t^7 + 6t^6 + 11t^5 + 12t^4 + 11t^3 + 6t^2 + 2t + 1$$

$$t^8 + 3t^7 + 10t^6 + 21t^5 + 26t^4 + 21t^3 + 10t^2 + 3t + 1$$

A closed form formula for the mod 2 Poincaré series can be derived from the formula produced by Liu and Schaffhauser [11] section 6.2.

## 7. ORIENTABILITY AND MONOTONICITY

**Proposition 7.1.** *Let  $(\Sigma, \tau)$  be a real curve with Real line bundle  $\xi$ . Then every element of the Picard group  $\text{Pic}(M(r, \xi)) \cong \mathbb{Z}$  can be represented by a Cartier divisor  $D$  such that  $\tau(D) = D$ .*

*Proof.* It was proven by Drezet and Narasimhan ([8], Theorem B) that  $\text{Pic}(M(r, \xi)) \cong \mathbb{Z}$  and is generated by the *theta divisor*  $\Theta$  constructed as follows. Choose a fixed semistable, algebraic vector bundle  $\mathcal{F}$  over  $\Sigma$  (of some particular rank and degree which is unimportant for our purposes) and define

$$\Theta := \{\mathcal{E} \in M(r, \xi) \mid H^0(\Sigma, \mathcal{E} \otimes \mathcal{F}) \neq 0\}.$$

Since  $\Theta$  is independent of  $\mathcal{F}$ , so  $\tau(\Theta) = \Theta$ . Since every other element of  $\text{Pic}(M(r, \xi))$  may be represented by  $k\Theta$  for some  $k \in \mathbb{Z}$ , the result follows.  $\square$

**Corollary 7.2.** *Let  $(\Sigma, \tau)$  be a real curve with Real line bundle  $\xi$ . The dualizing sheaf  $\omega$  of  $M(r, \xi)$  is a Real line bundle with fixed point set  $\omega^\tau$  a topologically trivial  $\mathbb{R}^1$ -bundle over  $M(r, \xi)^\tau$ . In particular, if  $M(r, \xi)^\tau$  is non-singular, then  $M(r, \xi)^\tau$  is an orientable manifold.*

*Proof.* Drezet and Narasimhan ([8], Theorem F) prove that the dualizing sheaf  $\omega$  on  $M(r, \xi)$  is equal to  $\omega = \mathcal{O}(-2n\Theta) = \mathcal{L}^{\otimes 2}$ , where  $\mathcal{L} = \mathcal{O}(2n\Theta)$ . By Lemma 7.1  $\mathcal{L}$  is real, so  $\omega^\tau \cong (\mathcal{L}^\tau)^{\otimes 2}$  is trivial.  $\square$

Assume now that  $\gcd(r, d) = 1$ . With the standard symplectic structure  $(M(r, \xi), \Omega)$  is monotone with  $c_1(TM(r, \xi)) = 2[\Omega]$  in  $H^2(M(r, \xi), \mathbb{Z}) \cong \mathbb{Z}$ . We have the following easy consequence.

**Proposition 7.3.** *If  $\gcd(r, d) = 1$  then  $M(r, \xi)^\tau$  is a monotone Lagrangian submanifold of  $(M(r, \xi), \Omega)$  with minimal Maslov number a positive multiple of 2.*

*Proof.* The Maslov index of a disk  $D \rightarrow M$  bounding a Real Lagrangian is given simply given by  $c_1(TM|_{D \cup \tau D})$  ([13] Theorem C.3.6). In our case, since  $c_1(TM) = 2[\Omega]$  is twice another integral class, the minimal Maslov number for disks must be a multiple of 2.  $\square$

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